# Singularities of Mappings São Carlos 2012 

David Mond

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## 1 Preface

These lecture notes accompany the series of four lectures on singularities of mappings given at the July 2012 summer school in the ICMC São Carlos. The lectures will cover only part of the material of the lecture notes. The plan is to speak on

1. Germs and the local conical structure. Milnor fibres as paradigm. Left-right equivalence and $A_{e}$ tangent space. Geometric criterion for finite codimension, leading to notion of stable perturbation. Examples of map-germs and calculation of their tangent space. Versality criterion and examples of versal deformations and stable perturbations. Conservation of multiplicity. Exercises.
2. Finite determinacy, finite codimension. The discriminant of a stable map-germ is a free divisor. Liftable and lowerable vector fields. Image and discriminant Milnor number vs $\mathcal{A}_{e}$-codimension. Open questions: relation of $\mathcal{A}_{e}$-codimension and image Milnor number, Lê-Ramanujam-type conjecture.
3. Multiple point spaces. Alternating cohomology. Spectral sequence for the homology of the image. Houston's theorem on the collapse of the spectral sequence. Open questions.
4. Calculation of presentation. Fitting ideals. Calculations. Open questions: are Fitting quotients Cohen-Macaulay? Lê conjecture.

The contents of the lectures and lecture notes reflect my own interests and knowledge more than a measured judgement in what is important in the subject. I am aware of omitting vast areas of wonderful mathematics, and of lamentable incompleteness even in what I have attempted to cover.

I am very grateful to the organisers for the opportunity to speak on this subject and for the stimulus of preparing these lecture notes. São Carlos has become a key centre for singularity theory, and it is always a pleasure to return here.

## 2 Introduction

The crucial notion is of course the derivative of a smooth or analytic mapping: if $f: X \rightarrow Y$ is a map of manifolds and $x \in X$ then $d_{x} f: T_{x} X \rightarrow T_{f(x)} Y$ is the derivative, defined by

$$
d_{x} f(\hat{x})=\lim _{h \rightarrow 0} \frac{f(x+h \hat{x})-f(x)}{h}
$$

if $X$ and $Y$ are open sets in linear spaces. If $X$ and $Y$ are contained, but not open, in linear spaces, $d_{x} f$ can be defined by restricting to $T_{x} X$ the derivative of a suitable extension of $f$ to an open set in the linear ambient space; otherwise one uses charts. It is also worth recalling that every tangent vector $\hat{x} \in T_{x} X$ is the tangent vector $\gamma^{\prime}(0)$ to a parameterised curve $\gamma:(\mathbb{R}, 0) \rightarrow(X, x)$ (or $\gamma:(\mathbb{C}, 0) \rightarrow(X, x)$ in the complex analytic category), and that $d_{x} f$ satisfies

$$
\begin{equation*}
d_{x} f\left(\gamma^{\prime}(0)\right)=(f \circ \gamma)^{\prime}(0) \tag{2.1}
\end{equation*}
$$

This may be taken as the definition. It is particularly useful in infinite dimensional cases, such as where $X$ is a group of diffeomorphisms.

A point $x \in X$ is a regular point of $f$ if $d_{x} f$ is surjective, and a critical point if it is not. The image of a critical point is a critical value of $f$; any point in $Y$ which is not a critical value is a regular value (even if it has no preimages). The set of all critical values is often called the discriminant of the map $f$. If $x_{0}$ is a regular point then $f$ is said to be a submersion at $x_{0}$. If $x_{0}$ is a regular point, then a simple argument based on the inverse function theorem establishes

Theorem 2.1. (Normal form for submersions) Suppose that $\operatorname{dim} X=n \geq k=\operatorname{dim} Y$ and $x_{0}$ is a regular point of $f: X \rightarrow Y$. Then one can choose coordinates $x_{1}, \ldots, x_{n}$ on $X$ around $x_{0}$, and $y_{1}, \ldots, y_{k}$ on $Y$ around $f\left(x_{0}\right)$, such that $f$ takes the form $f\left(x_{1}, \ldots, x_{n}\right)=\left(x_{1}, \ldots, x_{k}\right)$.

These notions are only of interest when $\operatorname{dim} X \geq \operatorname{dim} Y$; when $\operatorname{dim} X<\operatorname{dim} Y$, all points of $X$ are critical points, and the set of critical values of $f$ is the whole image of $f$. In this case one is interested in whether or not $d_{x} f$ is injective. If it is, $f$ is an immersion at $x_{0}$, and one has

Theorem 2.2. (Normal form for immersions) Suppose that $\operatorname{dim} X=n \leq k=\operatorname{dim} Y$ and that $f: X \rightarrow Y$ is an immersion at $x_{0}$. Then one can choose coordinates around $x_{0}$ and $f\left(x_{0}\right)$ such that $f$ takes the form $f\left(x_{1}, \ldots, x_{n}\right)=\left(x_{1}, \ldots, x_{n}, 0, \ldots, 0\right)$.

Exercise 2.3. 1. Find proofs of 2.1 and 2.2. Both follow from the inverse function theorem, by incorporating $f$ into a suitable auxiliary mapping whose derivative is invertible.
2. Prove that if $f:\left(k^{n}, 0\right) \rightarrow\left(k^{p}, 0\right)$ has rank $k$ at 0 then in suitable coordinates $f$ takes the form

$$
\left(x_{1}, \ldots, x_{n}\right) \rightarrow\left(x_{1}, \ldots, x_{k}, f_{k+1}(x), \ldots, f_{p}(x)\right) .
$$

Singularity theory begins where these two theorems end: it is concerned with what happens at points where $f$ is neither a submersions nor an immersion. It concentrates on the local behaviour of mappings, and for this reason uses the notion of germ of mapping, which we study briefly in Subection 2.1. Geometrical singularity theory for the two cases $\operatorname{dim} X \geq \operatorname{dim} Y$ and $\operatorname{dim} X<\operatorname{dim} Y$ is rather different. In the first case, classical singularity theory is interested in preimages $f^{-1}\left(y_{0}\right)$, and there is also a theory of the discriminant, initiated by Teissier in [46]. In the second case, to which much less attention has been devoted, one studies the images of maps. In fact very little is known about the geometry of maps in case $\operatorname{dim} X<\operatorname{dim} Y-1$, and the theory for the case $\operatorname{dim} X=\operatorname{dim} Y-1$ has an embarassing gap, in the form of an unproved (and unrefuted) conjecture which I made twenty five years ago.

This minicourse will concentrate on two key invariants for singularities of mappings, and the relation between them. The first comes from deformation theory: it is the deformation-theoretic codimension, and is the subject of Section . Until then, one can use the following relatively nontechnical working definition: it is the minimal number of parameters for a family of mappings in which a singularity equivalent to the one in question occurs 'stably' or 'irremovably'. The second, studied in Section 3, comes from topology: it is the "rank of the vanishing homology (of a nearby stable object)". This vague phrase will be made more precise; for now, we make do with two examples. The first is the non-degenerate critical point of a polynomial or analytic function, equivalent, by the Morse Lemma, to the germ defined by

$$
f\left(x_{1}, \ldots, x_{n}\right)=x_{1}^{2}+\cdots+x_{n}^{2} .
$$

Here $f^{-1}(0)$ is contractible, but for $t \neq 0, f^{-1}(t)$ has the homotopy-type of an $n$-sphere ${ }^{1}$. When $t$ returns to 0 , the rank of the homology of $f^{-1}(t)$ diminishes by 1 ; this is the 'rank of the vanishing homology' for this example. The second is the three pieces of plane curve which meet at a point in the Reidemeister move of type III. This configuration is evidently unstable: one can move any one of the three to form a triangle. Since now all intersections are transverse, this configuration is stable. It is the 'nearby stable object' for this example, and its vanishing homology, generated by the 1-cycle highlighted in the drawing on the right, once again has rank 1.


The deformation-theoretic codimension in the second example is also equal to 1 ; therein lies its importance in knot theory. Given two plane projections of the same knot, one can be deformed to the other in such a way that during the deformation, only three types of qualitative change occur. These are the three 'Reidemeister moves', and our example shows the third of these. They cannot be avoided in a 1-parameter family of projections; other more complicated singularities can be.

Notation and Terminology 2.4. Let $X$ and $Y$ be manifolds, and $f: X \rightarrow Y$ a differentiable map.

1. A singular point, or singularity of $f$ is a point where $f$ is not a submersion, in case $\operatorname{dim} X \geq$ $\operatorname{dim} Y$, and not an immersion, in case $\operatorname{dim} X \leq \operatorname{dim} Y$.
2. A map $X \rightarrow Y$ has corank $r$ at $x_{0}$ if the rank of $d_{x_{0}} f$ is $r$ less than the maximum possible, $\min \{\operatorname{dim} X, \operatorname{dim} Y\}$. Thus if $\operatorname{dim} X \leq \operatorname{dim} Y$ then $f$ has corank $r$ at $x_{0}$ if $r$ is the dimension of the kernel of $d_{x_{0}} f$, and if $\operatorname{dim} X \geq \operatorname{dim} Y$ then the corank is the dimension of the cokernel of $d_{x_{0}} f$.
3. If $Z \subset X$ then a singular point of $Z$ is a point at which $Z$ is not a submanifold of $X$.

### 2.1 Germs, cones and local rings

Definition 2.5. Let $f, g: X \rightarrow Y$ be maps of topological spaces, and let $S \subset X$.

1. We say that $f$ and $g$ have the same germ at $S$ (or along $S$ if $S$ is not a finite point set), if there is a neighbourhood $U$ of $S$ in $X$ such that $f$ and $g$ coincide on $U$. This is evidently an equivalence relation, and a germ of mapping at $S$ is an equivalence class under this relation.

[^0]2. Two subsets $X_{1}$ and $X_{2}$ of $X$ have the same germ at (or along) $S$ if there is a neighbourhood $U$ of $S$ in $X$ such that $X_{1} \cap U=X_{2} \cap U$. A germ at $S$ of subset of $X$ is an equivalence class of subset under this relation.
We denote a germ at $S$ of mapping $X \rightarrow Y$ by $f:(X, S) \rightarrow Y$, or $f:(X, S) \rightarrow(Y, T)$ if $f(S) \subset T \subset Y$. To determine a germ of mapping at $S$, it is enough to specify the behaviour of $f$ on some neighbourhood of $S$ in $X$. Usually $X$ is $\mathbb{C}^{n}$ or an analytic variety embedded in $\mathbb{C}^{n}, S$ is a single point or a finite set, and we specify $f$ by means of power series which converge in some neighbouhood of the points of $S$. Not every power series can be extended to a globally defined map $X \rightarrow Y$, so really our subject is not 'germs at $S$ of maps $X \rightarrow Y^{\prime}$, but 'germs at $S$ of maps to $Y$ from some neighbourhood of $S^{\prime}$. In practice this will not cause any difficulty.

Germs of maps to $\mathbb{C}$ can be added and multiplied, and the set of germs at $x_{0}$ of analytic functions on $X$ is a $\mathbb{C}$-algebra. It is denoted $\mathcal{O}_{X, x_{0}}$.

The notion of germ is particularly natural in the complex-analytic category, because of uniqueness of analytic continuation: if $U_{1}$ and $U_{2}$ are connected open sets in $\mathbb{C}^{n}$ and $f_{i}: U_{i} \rightarrow \mathbb{C}^{p}$ are complex analytic maps, then if $f_{1}$ and $f_{2}$ coincide on some open $V \subset U_{1} \cap U_{2}$, they coincide on all of $U_{1} \cap U_{2}$.

Exercise 2.6. Show that the same is not true of real $C^{\infty}$ maps.
If $X$ and $Y$ are spaces, and we select some class of germs of maps $X \rightarrow Y$ - e.g. germs of continuous maps, or germs of complex analytic maps in case $X$ and $Y$ are complex analytic varieties - then we can put together all of the germs into a global object, a sheaf. This notion is crucial in algebraic and analytic geometry, but I do not want to make it a prerequisite for this course, and certainly do not have time to develop it in detail here. Instead I will attempt to give a working definition sufficient to make some of the necessary theorems at least vaguely comprehensible, and urge the reader to find the time to study elsewhere.

The definition of sheaf requires an algebraic structure, so we take, as our target space $Y$, the field $\mathbb{C}$. It is natural to associate to each open $U \subset X$ the set

$$
\mathcal{O}_{X}(U):=\{f: U \rightarrow \mathbb{C}: f \text { is complex analytic }\}
$$

and make it into a $\mathbb{C}$-algebra by defining the operations pointwise

$$
(f+g)(x)=f(x)+g(x), \quad(f g)(x)=f(x) g(x), \quad(\lambda f)(x)=\lambda f(x) \text { for } \lambda \in \mathbb{C} .
$$

If $U \subset V$ there is a restriction map $\rho_{U, V}: \mathcal{O}(V) \rightarrow \mathcal{O}(U)$ which is a homomorphism of $\mathbb{C}$-algebras, and if $U \subset V \subset W$ then evidently

$$
\begin{equation*}
\rho_{U, V} \circ \rho_{V, W}=\rho_{U, W} . \tag{2.2}
\end{equation*}
$$

Let $\mathcal{U}_{x}$ be the collection of all neighbourhoods of a point $x$. The equivalence relation by which we arrived at the notion of germ of function or mapping becomes a relation on the disjoint union $\coprod_{U \in \mathcal{U}_{x}} \mathcal{O}(U):$
$f \in \mathcal{O}(U)$ and $g \in \mathcal{O}(V)$ are equivalent if there exists $W \in \mathcal{U}_{x}$ such that $\rho_{W, U}(f)=\rho_{W, V}(g)$.
The set of equivalence classes, $\mathcal{O}_{X, x_{0}}$, is in a natural way a $\mathbb{C}$-algebra: if $f, g \in \mathcal{O}_{X, x_{0}}$ then they can be represented by some $f_{1} \in \mathcal{O}(U)$ and $g_{1} \in \mathcal{O}(V)$, for some open neighbourhoods $U, V$ of $x_{0}$, and then the restrictions $\rho_{U \cap V, U}(f)$ and $\rho_{U \cap V, V}(g)$ in $U \cap V$ can be added or multiplied in the usual way. The sum and product of these restrictions then determine germs at $x_{0}$, which, as one can easily check, are independent of the choices of representative $f_{1}, g_{1}$.

Exercise 2.7. Show this.
The map $\rho_{x_{0}, U}: \mathcal{O}(U) \rightarrow \mathcal{O}_{X, x_{0}}$ defined by sending $f \in \mathcal{O}(U)$ to its germ at $x_{0}$ is a $\mathbb{C}$-algebra homomorphism. Evidently

$$
\rho_{x_{0}, V}=\rho_{x_{0}, U} \circ \rho_{U, V} .
$$

## Exercise 2.8. Is $\rho_{x_{0}, U}$ surjective? Injective?

The procedure we have outlined can be applied equally well to functions of other types: continuous, or $C^{\infty}$, or real analytic, etc. It also makes sense in a wider context:

Exercise 2.9. Let $f: X \rightarrow Y$ be a map of topological spaces. For $U \subset X$ define $\mathcal{H}^{p}(U):=$ $H^{p}\left(f^{-1}(U)\right.$ ) (the $p$-th topological cohomology of $f^{-1}(U)$ ).

1. Given $U \subset V \subset X$, show how to define $\rho_{U, V}: \mathcal{H}^{p}(V) \rightarrow \mathcal{H}^{p}(U)$ so that (2.2) holds.
2. Show that if $f$ is a locally trivial fibre bundle then for $U \in \mathcal{U}_{x}$ sufficiently small and contractible, $\mathcal{H}^{p}(U) \simeq \mathcal{H}^{p}(\{x\})$.

A further justification for the use of the notion of germ in singularity theory comes from the fact that closed analytic spaces are 'locally conical'. This is particularly important in the definition of the vanishing homology, so we go into some detail here. If $X$ is any topological space, the cone on $X$, which we denote by $C(X)$, is obtained by forming the Cartesian product $X \times[0,1]$ and then identifying all of the points of $X \times\{1\}$ with one another. One writes $C(X)=(X \times[0,1]) /(X \times\{1\})$, where the notation $B / A$, for $A$ a subset of $B$, means the quotient of $B$ by the equivalence relation which identifies all the points of $A$ to one another.


Exercise 2.10. For any space $X, C(X)$ can be contracted to its vertex.
Because cones are contractible, their homology is equal to that of a point.
Theorem 2.11. Let $U \subset \mathbb{C}^{n}$ be open and let $X \subset U$ be the set of common zeros of $k$ analytic functions $f_{1}, \ldots, f_{k} \in \mathcal{O}(U)$. If $x_{0} \in X$, there exists $\varepsilon>0$ such that $X \cap B_{\varepsilon}\left(x_{0}\right)$ is homeomorphic to the cone on its boundary $X \cap S_{\varepsilon}\left(x_{0}\right)$.

Exercise 2.12. Show that this is true if $X=\mathbb{C}^{n}$, and therefore if $X$ is a smooth manifold at $x_{0}$.
Write $X_{\varepsilon}:=S_{\varepsilon}\left(x_{0}\right) \cap X$ and $X_{\leq \varepsilon}:=X \cap B_{\varepsilon}\left(x_{0}\right)$. If $X$ is a $k$-dimensional manifold except at $x_{0}$ (i.e. $X$ has isolated singularity at $x_{0}$ ) then the theorem can be proved by

1. constructing a 'radial' vector field $v$, pointing in towards $x_{0}$, on a neighbourhood of $x_{0}$ in $X$, and adjusting the length of the vectors so that for each point $x \in X_{\varepsilon}$, the trajectory $\varphi_{t}(x)$ starting at $x$ arrives at $x_{0}$ at time $t=1$, and
2. defining a homeomorphism $H: X_{\varepsilon} \times[0,1) \rightarrow X_{\leq \varepsilon} \backslash\left\{x_{0}\right\}$ by

$$
H(x, t)=\varphi_{t}(x),
$$

3. which (automatically) extends to a homeomorphism $\left(X_{\varepsilon} \times[0,1]\right) /\left(X_{\varepsilon} \times\{1\}\right) \rightarrow X_{\leq \varepsilon}$.

The theorem holds also for locally closed real analytic subsets of $\mathbb{R}^{n}$ with isolated singularities, but not in general for the zero loci of $C^{\infty}$ functions. A more involved argument, using Whitney regular stratifications, proves the theorem for the case where $X$ is a (real or complex) analytic set with arbitrary singularity at $x_{0}-$ see [4].

Exercise 2.13. Give an example to show that the zero-loci of $C^{\infty}$ functions need not be locally conical.

Exercise 2.14. Suppose that $X$ has isolated singularity at 0 , and that there is a function $\rho$ : $X \rightarrow \mathbb{R}_{\geq 0}$ such that

1. $\rho$ has no critical point in $X_{\leq \varepsilon} \backslash\left\{x_{0}\right\}$, and
2. $\rho^{-1}(0)=\left\{x_{0}\right\}$.

Use the gradient vector of $\rho$ to construct the vector field of the sketched proof of 2.11.
The hardest part of the proof of 2.11 comes in showing that such a function exists. In fact any real analytic function $\rho: X \rightarrow \mathbb{R}_{\geq 0}$ satisfying $2.14(2)$ will do; one uses the curve selection lemma ( $c f[35]$ ) to show that it also satisfies $2.14(1)$ for some $\varepsilon>0$. In particular, one can use the Euclidean distance-squared function $\rho_{E}(x):=\left\|x-x_{0}\right\|^{2}$.

Exercise 2.15. Show that $\rho_{E}$ satisfies 2.14(1) iff for all $\varepsilon^{\prime}$ with $0<\varepsilon^{\prime} \leq \varepsilon, X \pitchfork S_{\varepsilon^{\prime}}\left(x_{0}\right)$.
Exercise 2.16. Divide up the objects pictured below into subsets which are cones on their boundary.


Planar projection of a knot


Exercise 2.17. What is the appropriate version of locally conical structure for a mapping? (I am not sure there is a right answer here).

The local conical structure is crucially important in singularity theory. It gives a clear meaning to the term "local", and it makes possible the idea of local changes in a deformation. The simplest example along these lines is the Milnor fibre of an isolated hypersurface singularity. We have already seen that if $f$ is an analytic function on some open set in $\mathbb{C}^{n}$ and has isolated singularity at $x_{0}$, then there exists $\varepsilon>0$ such that $B_{\varepsilon}\left(x_{0}\right) \subset U$ and $f^{-1}\left(y_{0}\right) \cap B_{\varepsilon}\left(x_{0}\right)$ is homeomorphic to the cone on $f^{-1}\left(y_{0}\right) \cap S_{\varepsilon}\left(x_{0}\right)$ - indeed, that $f^{-1}\left(y_{0}\right) \pitchfork S_{\varepsilon^{\prime}}\left(x_{0}\right)$ for all $\varepsilon^{\prime}$ with $0<\varepsilon^{\prime} \leq \varepsilon$. An argument involving properness shows also that

Proposition 2.18. In this case, there exists $\eta>0$ (depending on the choice of $\varepsilon$ ) such that provided $\left|y_{0}-y\right| \leq \eta$ then $f^{-1}(y) \pitchfork S_{\varepsilon}\left(x_{0}\right)$. For such $\varepsilon$ and $\eta$, the map

$$
f \mid: B_{\varepsilon}\left(x_{0}\right) \cap f^{-1}\left(B_{\eta}^{*}\left(y_{0}\right)\right) \rightarrow B_{\eta}^{*}\left(y_{0}\right)
$$

is a locally trivial fibre bundle.
The same principle gives us the notion of the "nearby stable object" (near to a singularity with isolated instability) in other situations. The details may be more complicated but the basic idea is the same.

### 2.2 Background in commutative algebra

If $X$ is any analytic space and $p \in X$, then the evaluation map

$$
\mathcal{O}_{X, p} \rightarrow \mathbb{C}, \quad f \mapsto f(p)
$$

is surjective, so that its image is the field $\mathbb{C}$. Its kernel is therefore a maximal ideal in $\mathcal{O}_{X, p}$, which is denoted by $\mathfrak{m}_{X, p}$. Indeed it is the only maximal ideal, since if $f \in \mathcal{O}_{X, p}$ is not in $\mathfrak{m}_{X, p}$ then $1 / f \in \mathcal{O}_{X, p}$, so that any ideal containing $f$ also contains 1 and therefore all of $\mathcal{O}_{X, p}$. This shows that every proper ideal of $\mathcal{O}_{X, p}$ is contained in $\mathfrak{m}_{X, p}$. Rings with a single maximal ideal are called local rings. Their properties play a very large rôle in singularity theory.

We will frequently abbreviate $\mathfrak{m}_{X, p}$ simply to $\mathfrak{m}$. If $x_{1}, \ldots x_{n}$ are coordinates on $X$ around $p$, and $p=\left(p_{1}, \ldots, p_{n}\right)$ in these coordinates, then every germ $f \in \mathcal{O}_{X, p}$ can be written as a convergent power series in $x_{1}-p_{1}, \ldots, x_{n}-p_{n}$. It follows that

$$
\begin{equation*}
\mathfrak{m}_{X, p}=\left(x_{1}-p_{1}, \ldots, x_{n}-p_{n}\right) \tag{2.4}
\end{equation*}
$$

(the ideal generated by $x_{1}-p_{1}, \ldots, x_{n}-p_{n}$ ).
In any ring $R$, the sum and product of ideals $I$ and $J$ are defined simply by

$$
I+J=\{r+s: r \in I, s \in J\}, \quad I J=\left\{\sum_{i=0}^{m} r_{i} s_{i}: m \in \mathbb{N}, r_{i} \in I, s_{i} \in J \text { for all } i\right\} .
$$

Exercise 2.19. 1. Show that in any ring $R$, if $I$ and $J$ are ideals then so are $I+J$ and $I J$.
2. Let $X=\mathbb{C}^{n}$ and $p=0$.
(a) Show that $\mathfrak{m}^{2}=\left\{f \in \mathcal{O}_{\mathbb{C}^{n}, 0}: f(0)=\partial f / \partial x_{i}(0)=0\right.$ for $\left.i=1, \ldots, n\right\}$.
(b) Show more generally that

$$
\mathfrak{m}^{k}=\left\{f \in \mathcal{O}_{\mathbb{C}^{n}, 0}: \partial^{\alpha} f / \partial x^{\alpha}(0)=0 \text { for } 0 \leq|\alpha| \leq k-1\right\}
$$

where $\alpha$ is a multi-index $\alpha=\left(\alpha_{1}, \ldots, \alpha_{n}\right),|\alpha|=\alpha_{1}+\cdots+\alpha_{n}$, and by $\partial^{0} f / \partial x^{0}$ we mean simply $f$.

In the $C^{\infty}$ category, (2.4) and 2.19(a) and (b) also hold. However (2.4) is no longer completely obvious, and is known as Hadamard's Lemma - see Martinet's book [27], Chapter 1.

We will make much use of the following statement.
Lemma 2.20. (Nakayama's Lemma) Let $M$ be a finitely generated module over a Noetherian local ring $R$ with maximal ideal $\mathfrak{m}$. If $\mathfrak{m} M=M$ then $M=0$.

Corollary 2.21. Let $M$ and $N$ be submodules of an $R$-module $P$, with $M$ finitely generated, and suppose that

$$
\begin{equation*}
M \subset N+\mathfrak{m} M \tag{2.5}
\end{equation*}
$$

Then $M \subset N$.
Proof Let $m_{1}, \ldots, m_{r}$ generate $M$ over $R$. Since $M=\mathfrak{m} M$, for each $i$ there exist $\alpha_{i j} \in \mathfrak{m}$ such that for $i=1, \ldots, r$,

$$
m_{i}=\alpha_{11} m_{1}+\cdots+\alpha_{1 r} m_{r}
$$

Rewriting these $r$ equations as a single matrix equation we get

$$
\left(\begin{array}{c}
m_{1} \\
\vdots \\
m_{n}
\end{array}\right)=\left(\begin{array}{ccc}
\alpha_{11} & \cdots & \alpha_{n 1} \\
\vdots & \cdots & \vdots \\
\alpha_{1 n} & \cdots & \alpha_{n n}
\end{array}\right)\left(\begin{array}{c}
m_{1} \\
\vdots \\
m_{n}
\end{array}\right)
$$

and therefore

$$
\left(I_{n}-A\right)\left(\begin{array}{c}
m_{1} \\
\vdots \\
m_{n}
\end{array}\right)=0
$$

where $I_{n}$ is the $n \times n$ identity matrix and $A$ is the matrix $\left[\alpha_{i j}\right]$. Multiplying both sides by the matrix of cofactors of $I_{n}-A$ we deduce that

$$
\operatorname{det}\left[I_{n}-A\right] m_{i}=0
$$

for all $i$. But $\operatorname{det}\left[I_{n}-A\right]$ is a unit in the ring $R$, since it is equal to $1+\alpha$ for some $\alpha \in \mathfrak{m}$. Hence $m_{i}=0$ for $i=1, \ldots, r$, and so $M=0$.

Proof of Corollary Let $M_{0}=(M+N) / N$. The hypothesis $M \subset N+\mathfrak{m} M$ implies that $M_{0}=$ $\mathfrak{m} M_{0}$. It follows by the Lemma that $M_{0}=0$, so that $M \subset N$.

### 2.3 Conservation of multiplicity

Suppose that $U$ is open in $\mathbb{C}^{n}$, that $f: U \rightarrow \mathbb{C}^{n}$ is analytic, that $f(a)=b$, and that $a$ is isolated in $f^{-1}(b)$ - that is, there exists $\varepsilon>0$ such that $f^{-1}(b) \cap B_{\varepsilon}(a)=\{a\}$. Then the ideal $f^{*} \mathfrak{m}_{\mathbb{C}^{n}, b}:=$ $\left(f_{1}-b_{1}, \ldots, f_{n}-b_{n}\right)$ must contain a power of the maximal ideal $\mathfrak{m}_{\mathbb{C}^{n}, a}$, since $\sqrt{f^{*} \mathfrak{m}_{\mathbb{C}^{n}, b}}=\mathfrak{m}_{\mathbb{C}^{n}, a}$. In fact

Proposition 2.22. The following three statements are equivalent:

1. $a$ is isolated in $f^{-1}(b)$;
2. $\operatorname{dim}_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^{n}, a} / f^{*} \mathfrak{m}_{\mathbb{C}^{n}, b}<\infty$;
3. $f^{*} \mathfrak{m}_{\mathbb{C}^{n}, b} \supset \mathfrak{m}^{k}$ for some $k<\infty$.

Proof. That 3 implies 2 implies 1 is obvious. The converse follows from Ruckert's Nullstellensatz: that for any ideal $I \subset \mathcal{O}_{\mathbb{C}^{n}, a}$, the ideal of all functions vanishing on $V(I)$ is the radical $\sqrt{I}:=$ $\left\{f \in \mathcal{O}_{\mathbb{C}^{n}, a}: f^{k} \in I\right.$ for some $\left.k\right\}$. Since each coordinate function $x_{i}-a_{i}$ vanishes on $V\left(f^{*} \mathfrak{m}_{\mathbb{C}^{n}, b}\right)$ it follows that $\left(x_{i}-a_{i}\right)^{k_{i}} \in f^{*} \mathfrak{m}_{\mathbb{C}^{n}, b}$ for some $k_{i}$. Then 3 holds with $k=n \max _{i}\left\{k_{i}\right\}-1$.

Exercise 2.23. Show that if $I$ is any ideal in $\mathcal{O}_{\mathbb{C}^{n}, x_{0}}$ such that $\operatorname{dim}_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^{n}, x_{0}} / I=k<\infty$ then $I \supset \mathfrak{m}^{k}$.

The dimension of $\mathcal{O}_{\mathbb{C}^{n}, a} / f^{*} \mathfrak{m}_{\mathbb{C}^{n}, b}$ is the multiplicity of $f$ at $a$; we will denote it by $\operatorname{mult}_{a}(f)$.
Theorem 2.24. Let $U$ be open in $\mathbb{C}^{n}$, let $f: U \rightarrow \mathbb{C}^{n}$ be analytic, and let $x_{0}$ be isolated in $f^{-1}\left(y_{0}\right)$. Then there exists $\varepsilon>0$ and $\eta>0$ such that for all $y \in B_{\eta}\left(y_{0}\right)$,

$$
\begin{equation*}
\sum_{x \in f^{-1}(y) \cap B_{\varepsilon}\left(x_{0}\right)} \operatorname{mult}_{x}(f)=\operatorname{mult}_{x_{0}} f . \tag{2.6}
\end{equation*}
$$

The equality (2.6) is the basis for a number of statements about conservation of multiplicity. Here are some examples.

Conservation of Milnor number: If $U$ is open in $\mathbb{C}^{n}$ and $f: U \rightarrow \mathbb{C}$ has isolated singularity at $x_{0}$ then the Milnor number of $f$ at $x_{0}$ is defined to be mult $x_{x_{0}}(d f)$ where $j^{1} f:\left(\mathbb{C}^{n}, x_{0}\right) \rightarrow\left(\mathbb{C}^{n}, 0\right)$ is the map with component functions $\partial f / \partial x_{1}, \ldots, \partial f / \partial x_{n}$. That is,

$$
\mu_{x_{0}}(f)=\operatorname{dim} \mathcal{O}_{\mathbb{C}^{n}, x_{0}} / J_{f}
$$

where $J_{f}$ is the jacobian ideal $\left(\partial f / \partial x_{1}, \ldots, \partial f / \partial x_{n}\right)$.
Corollary 2.25. Let $U$ be open in $\mathbb{C}^{n}$ and let $f: U \rightarrow \mathbb{C}$ have isolated singularity at $x_{0}$ with Milnor number $\mu<\infty$. Then in any deformation $F: U \times \mathbb{C}^{d} \rightarrow \mathbb{C}$ of $f$, there exists $\varepsilon>0$ and $\eta>0$ such that for $|u|<\eta$,

$$
\sum_{x \in B_{\varepsilon}\left(x_{0}\right)} \mu_{x}\left(f_{u}\right)=\mu_{x_{0}}(f) .
$$

Proof. Suppose first that the set

$$
S_{F}^{\mathrm{rel}}:=\left\{(x, u): \partial F / \partial x_{1}=\cdots=\partial F / \partial x_{n}=0 \text { at }(x, u)\right\}
$$

is smooth. Its dimension is necessarily equal to that of $U$, since $j^{1} f$ must be a submersion outside $x_{0}$.

Let $\pi: S_{F}^{\mathrm{rel}} \rightarrow U$ be projection. Since $S_{F}^{\mathrm{rel}}$ is locally isomorphic to $\mathbb{C}^{\operatorname{dim} U}$, we can apply 2.24 to the map $\pi$. If $(u, x) \in S_{F}^{\text {rel }}$ then

$$
\begin{equation*}
\mathcal{O}_{S_{F}^{\mathrm{rel},(u, x)}} / \pi^{*} \mathfrak{m}_{U,(v, u)} \simeq \mathcal{O}_{\mathbb{C}^{n}, x} / J_{f_{u}} \tag{2.7}
\end{equation*}
$$

and thus

$$
\operatorname{mult}_{(u, x)}(\pi)=\mu_{x} f_{u} .
$$

It follows from 2.24 that there exists $\varepsilon>0$ and $\eta>0$ such that for $|u|<\eta$,

$$
\sum_{x \in B_{\varepsilon}\left(x_{0}\right)} \mu_{x_{0}}\left(f_{u}\right)=\mu_{x_{0}}(f) .
$$

If $S_{F}^{\mathrm{rel}}$ is not smooth, one can further deform $F$ by a deformation $G: U \times \mathbb{C}^{d} \times \mathbb{C}^{e}$ such that $S_{G}^{\mathrm{rel}}$ is smooth of the requisite dimension - for example $G(x, u, v)=F(u, x)+\sum_{i} v_{i} x_{i}$. The first part of the argument applies to $G$, and the conclusion is obtained by restricting to $\{v=0\}$.

Exercise 2.26. 1. Prove the equality (2.7).
2. Show that if $S_{F}^{\mathrm{rel}}$ is smooth then $u$ is a regular value of $\pi$ if and only if $f_{u}$ has only nondegenerate critical points.

Conservation of intersection number of plane curves: If $C=\{f=0\}$ and $D=\{g=0\}$ are plane analytic curves meeting at $x_{0}$, their intersection number at $x_{0}, I_{x_{0}}(C, D)$, is defined to be the multiplicity at $x_{0}$ of the map $(f, g)$.

Corollary 2.27. Suppose the two curves $C$ and $D$ meet at $x_{0}$ with $I_{x_{0}}(C, D)<\infty$, and let $C_{t}$ and $D_{t}$ be parameterised families of plane curves with $C_{0}=C, D_{0}=D$. Then there exist $\varepsilon>0$ and $\eta>0$ such that for $|t|<\eta$,

$$
\sum_{x \in C_{t} \cap D_{t} \cap B_{\varepsilon}\left(x_{0}\right)} I_{x}\left(C_{t}, D_{t}\right)=I_{x_{0}}(C, D) .
$$

## Proof. Exercise

Conservation of cross-cap number: Suppose $f:\left(\mathbb{C}^{2}, 0\right) \rightarrow\left(\mathbb{C}^{3}, 0\right)$ is given by $f(x, y)=$ $\left(x, f_{2}(x, y), f_{3}(x, y)\right)$. Its non-immersve locus $S_{f}$ is determined by the equations $\partial f_{2} / \partial y=\partial f_{3} / \partial y=$ 0 . Suppose this set consists just of 0 . We define the cross-cap number of $f, C_{0}(f)$, as mult ${ }_{0}\left(\partial f_{2} / \partial y, \partial f_{3} / \partial y\right)$.

Exercise 2.28. 1. Find $C_{0}(f)$ in each of the following cases:
(a) $f(x, y)=\left(x, y^{2}, x y\right)$ (this is the paranetrisation of the Whitney umbrella, and is known as the cross-cap);
(b) $f(x, y)=\left(x, y^{2}, y^{3}+x^{k+1} y\right)$
(c) $f(x, y)=\left(x, y^{3}, x y+y^{3 k-1}\right)$.
2. Suppose that $F(x, y, u)=\left(x, F_{2}(x, y, u), F_{3}(x, y, u), u\right)$ is an unfolding of $f$ with $u \in \mathbb{C}^{d}$, and for fixed $u$ let $f_{u}(x, y)=\left(x, F_{2}(x, y, u), F_{3}(x, y, u)\right)$. Let $S_{F}$ be the non-immersive locus of $F$, and consider the projection $\pi: S_{F} \rightarrow \mathbb{C}^{d}$. Show that
(a) It is possible to choose $F$ so that $S_{F}$ is smooth of codimension 2 in $\mathbb{C}^{2} \times \mathbb{C}^{d}$.
(b) In this case $\operatorname{mult}_{(x, y, u)}(\pi)=C_{(x, y)}\left(f_{u}\right)$.
(c) There exist $\varepsilon>0$ and $\eta>0$ such that for $|u|<\eta$,

$$
\sum_{(x, y) \in S_{f_{u} \cap B_{\varepsilon}(0)}} C_{(x, y)}\left(f_{u}\right)=C_{0}(f)
$$

(d) One can show that if $C(f)=1$ then $f$ is $\mathcal{A}$-equivalent to the cross-cap, the germ of 1 (a). Conclude that there exist deformations $f_{u}$ of $f$ with $C(f)$ cross-caps.
3. Suppose that $f:\left(\mathbb{C}^{n}, 0\right) \rightarrow\left(\mathbb{C}^{n+1}, 0\right)$ has corank 1 . Show that the ideal of $(n-1) \times(n-1)$ minor determinants of the matrix of $d f$ (the ramification ideal of $f, \mathcal{R}_{f}$ ) is generated by some two of these minors. Hint: do this first when $n=2$, where it's easier to see what is going on. How many generators does $\mathcal{R}_{f}$ need when $f$ has corank 2 ? corank 3 ?
We will see other applications of 2.24 to prove conservation of multiplicity of one kind or another. However 2.24 is not sufficient in all cases. In the examples we have just seen, we applied 2.24 to the projection $\pi$ from the singular or relative critical space $S_{F}$ of a deformation $F$, to the parameter space $\mathbb{C}^{d}$. This relied upon being able to choose $F$ such that $S_{F}$ is smooth. However there are situations where this is not possible. For example, the non-immersive locus of an unfolding $F(x, y, u)=\left(F_{1}(x, y, u), F_{2}(x, y, u), F_{3}(x, y, u), u\right)$ has equations

$$
\operatorname{det}\left|\begin{array}{ll}
\frac{\partial F_{1}}{\partial x} & \frac{\partial F_{1}}{\partial y}  \tag{2.8}\\
\frac{\partial F_{2}}{\partial x} & \frac{\partial F_{2}}{\partial y}
\end{array}\right|=\operatorname{det}\left|\begin{array}{cc}
\frac{\partial F_{1}}{\partial x} & \frac{\partial F_{1}}{\partial y} \\
\frac{\partial F_{3}}{\partial x} & \frac{\partial F_{3}}{\partial y}
\end{array}\right|=\operatorname{det}\left|\begin{array}{ll}
\frac{\partial F_{2}}{\partial x} & \frac{\partial F_{2}}{\partial y} \\
\frac{\partial F_{3}}{\partial x} & \frac{\partial F_{3}}{\partial y}
\end{array}\right|=0
$$

and if $F$ is an unfolding of a map-germ of corank 2, then all three determinants lie in the square of the maximal ideal, so that their locus of common zeroes is unavoidably singular.

Nevertheless, it is still true that, just as shown in Exercise 2.282(d) above, for a finitely determined map-germ $\left(\mathbb{C}^{2}, 0\right) \rightarrow\left(\mathbb{C}^{3}, 0\right)$, the number of cross-caps appearing in a stable perturbation is equal to

$$
\operatorname{dim}_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^{2}, 0} / \mathcal{R}_{f}
$$

where $\mathcal{R}_{f}$ is the ramification ideal of $f$, generated by the three $2 \times 2$ minors of the matrix of $d f$ (as for $F$ in (2.8) above). The proof of this makes use of the notion of Cohen-Macaulay rings and spaces, and involves some quite serious, though by now rather standardised, commutative algebra arguments. Instead of 2.24 we use
Theorem 2.29. Let $U$ be open in an n-dimensional Cohen Macaulay variety $X \subset \mathbb{C}^{N}$, let $f$ : $U \rightarrow \mathbb{C}^{n}$ be analytic, and let $x_{0}$ be isolated in $f^{-1}\left(y_{0}\right)$. Then there exists $\varepsilon>0$ and $\eta>0$ such that for all $y \in B_{\eta}\left(y_{0}\right)$,

$$
\begin{equation*}
\sum_{x \in f^{-1}(y) \cap B_{\varepsilon}\left(x_{0}\right) \cap X} \operatorname{mult}_{x}(f)=\text { mult }_{x_{0}} f . \tag{2.9}
\end{equation*}
$$

In the example described above, $V\left(\mathcal{R}_{f}\right)$ is Cohen Macaulay provided its codimension in the domain of the unfolding $F$ is equal to 2 . This is a consequence of Theorem 2.43 below.

The proofs of Theorems 2.24 and 2.29 run along the same lines. The first step is to show that $\mathcal{O}_{X, x_{0}}$ is a finitely generated module over $\mathcal{O}_{\mathbb{C}^{n}, 0}$. For this one uses the Preparation Theorem, 2.30 below. The second step is to use the Cohen-Macaulayness of $\mathcal{O}_{X}$ to show that it is not only finitely generated but free over $\mathcal{O}_{\mathbb{C}^{n}, 0}$.

### 2.4 The preparation theorem

The following theorem has rather an algebraic appearance, but is in fact a theorem of analysis. The classical Weierstrass Preparation Theorem on which it is based concerns division of analytic functions, and is more evidently "analytic".

Theorem 2.30. Let $X$ and $Y$ be complex manifolds (or, more generally, analytic spaces) and let $f:\left(X, x_{0}\right) \rightarrow\left(Y, y_{0}\right)$ be an analytic map germ. Let $M$ be a finitely generated module over $\mathcal{O}_{X, 0}$. The following statements are equivalent.

1. $M$ is also finitely generated over $\mathcal{O}_{Y, y_{0}}$ via $f$.
2. $\operatorname{dim}_{\mathbb{C}} M / f^{*} \mathfrak{m}_{Y, y_{0}} M<\infty$.

It is extensively used in analytic geometry and singularity theory. The statement also holds, verbatim, for $C^{\infty}$ mappings and modules over the ring $\mathscr{E}_{n}$ of $C^{\infty}$ germs. This much harder theorem was proved by Bernard Malgrange, at the urging of René Thom, in the 1960's, and made possible Thom's Catastrophe Theory, and Mather's celebrated series of papers on the stability of $C^{\infty}$ mappings, [28], [30], [29], [31], [32], [33]. Lojasiewicz and Mather himself published alternative proofs.

### 2.5 Jet spaces and jet bundles

We denote by $J^{k}(n, p)$ the space of $p$-tuples of polynomials of degree $\leq k$ in $n$ variables with no constant term. A map-germ : $\left(\mathbb{C}^{n}, 0\right) \rightarrow\left(\mathbb{C}^{p}, 0\right)$ determines a germ of map $j^{k} f:\left(\mathbb{C}^{n}, 0\right) \rightarrow J^{k}(n, p)$, the $k$-jet extension of $f$, defined by

$$
j^{k} f(x)=\text { degree } \mathrm{k} \text { Taylor polynomial of } f \text { at } x \text {, without its constant term. }
$$

The Taylor polynomial of $f$ is determined by partial derivatives of order $\leq k$ of the component functions of $f$ at $x$, so the $k$-jet can be thought of as simply recording these partial derivatives. There is a also a jet bundle $J^{k}(X, Y)$ over any pair of manifolds $X$ and $Y$, whose fibre over $\left(x_{0}, y_{0}\right) \in X \times Y$, which we denote by $J^{k}(X, Y)_{(x, y)}$, is the set of $k$-jets of germs of maps $\left(X, x_{0}\right) \rightarrow\left(Y, y_{0}\right)$. Two such map-germs determine the same $k$-jet at $x$ if they have the same partials of order $\leq k$ at $x$, with respect to some, and therefore to any, local coordinate systems on $X$ and $Y$. So in coordinate free terms, a $k$-jet is an equivalence class of map-germs $(X, x) \rightarrow(Y, y)$.

Although $J^{k}(n, p)$ is a vector space, the fibre $J^{k}(X, Y)_{\left(x_{0}, y_{0}\right)}$ is not; for the identifications between the two spaces depends on a choice of coordinate system, and when we change coordinates the higher derivatives of $f$ change in a non-linear way. Thus there is no natural way of providing $J^{k}(X, Y)_{\left(x_{0}, y_{0}\right)}$ with the operations of a vector space, and $J^{k}(X, Y)$ is not a vector bundle over $X \times Y$.

Nevertheless, $J^{k}(X, Y)$ is a locally trivial fibre bundle over $X \times Y$.
Its importance for us is because of its role as a kind of Platonic Heaven which houses ideal versions of all of the singularities which appear in mappings. I will spend the rest of this section justifying this metaphysical remark.

Consider first the 1-jet-bundle $J^{1}(N, P)$. By a choice of local coordinates on $U_{X} \subset X$ and $U_{Y} \subset Y$ we can identify $\pi^{-1}\left(U_{X} \times U_{Y}\right)$ with a product $V_{X} \times V_{Y} \times J^{1}(n, p)$ where $V_{X} \subset \mathbb{C}^{n}, Y \subset \mathbb{C}^{p}$ are open sets. The information contained in the 1-jet $j^{1} f(x)$ is just the values of the first order partials of $f$, so we can think of $j^{1} f$ as the map

$$
x \mapsto\left(x, f(x),\left[d_{x} f\right]\right) \in \mathbb{C}^{n} \times \mathbb{C}^{p} \times \operatorname{Mat}_{p \times n}(\mathbb{C})
$$

where $\left[d_{x} f\right]$ is the $n \times p$ jacobian matrix of $f$ at $x$. Let u suppose, to fix ideas, that $n \leq p$, and define $\Sigma^{k}(n, p)$ (or $\Sigma^{k}$ when the dimensions are clear from the context) to be the set of $p \times n$ complex matrices of kernel rank $k$.

Exercise 2.31. $\Sigma^{k}(n, p)$ is a submanifold of $M a t_{p \times n}(\mathbb{C})$ of codimension $k(p-n+k)$. The formula for the codimension can be recalled as follows: a $p \times n$ matrix of the form

$$
\left(\begin{array}{cc}
I_{n-k} & B \\
0 & D
\end{array}\right)
$$

has kernel rank $k$ if and only if $D=0$. The same is true if we have an invertible $(n-k) \times(n-k)$ matrix $A$ in place of $I_{n-k}$. A more general matrix

$$
\left(\begin{array}{ll}
A & B \\
C & D
\end{array}\right)
$$

in which $A$ is of size $(n-k) \times(n-k)$ and invertible can be brought to this form by left-multiplying by

$$
\left(\begin{array}{cc}
I_{n-k} & 0 \\
-C A^{-1} & I_{p-n+k}
\end{array}\right)
$$

The matrix is in $\Sigma^{k}$ if all entries in the transformed $D$ are equal to zero. This gives $(p-n+k) k$ equations.

Let $f: X \rightarrow Y$ be a mapping, and denote now by $\Sigma^{k}(f)$ the set of points in $X$ where $d_{x} f$ has kernel rank $k$. Then $\Sigma^{k}(f)=\left(j^{1} f\right)^{-1}\left(\Sigma^{k}\right)$. Note, incidentally, that if we change coordinates on $X$ then of course $j^{1} f$ also changes, but $\left(j^{1} f\right)^{-1}\left(\Sigma^{k}\right)$ is, evidently, unchanged. This is because $\Sigma^{k}$ has the important property that it is preserved by the action of coordinate changes on $X$ (or on $Y$ ).

Observation: suppose $x_{0} \in \Sigma^{k}(f)$ and $j^{1} f \pitchfork \Sigma^{k}$ at $x_{0}$. Then

- $\Sigma^{k}(f)$ is a smooth submanifold of $X$ of codimension $k(p-n+k)$.
- Slightly less obvious: for $\ell<k, j^{1} f \pitchfork \Sigma^{\ell}$ also.
- Indeed, writing $m_{0}:=j^{1} f\left(x_{0}\right)$, there is a local diffeomorphism of germs of filtered spaces

$$
\left(\operatorname{Mat}_{p \times n}, m_{0}\right) \supset\left(\overline{\Sigma^{1}}, m_{0}\right) \supset \cdots \supset\left(\overline{\Sigma^{k-1}}, m_{0}\right) \supset\left(\Sigma^{k}, m_{0}\right)
$$

and

$$
\left(\left(X, x_{0}\right) \supset\left(\overline{\Sigma^{1}(f)}, x_{0}\right) \supset \cdots \supset\left(\overline{\Sigma^{k-1}(f)}, x_{0}\right) \supset\left(\Sigma^{k}(f), x_{0}\right)\right) \times \text { smooth factor }
$$

The second statement is a consequence of the fact that the corresponding stratification

$$
\operatorname{Mat}_{p \times n}(\mathbb{C}) \supset\left(\Sigma^{1} \backslash \overline{\Sigma^{2}}\right) \supset \cdots \supset\left(\Sigma^{\ell} \backslash \overline{\Sigma^{\ell+1}}\right) \cdots
$$

is Whitney regular. We do not dwell on this now. The aim is simply to make clear that the transversality of $j^{1} f$ to certain submanifolds of the jet bundle $J^{k}(X, Y)$ gives us a lot of information about submanifolds (subsets) of $X$ determined by the geometry of $f$. The subsets that we are interested in are those which are preserved by the action of the group of diffeomorphisms of $X$ and $Y$ - the so-called left-right invariant subsets of $J^{k}(X, Y)$. The hypothesis on the transversality of $j^{1} f$ to $\Sigma^{k}$ that we invoked in our observation is motivated by the following statement.

Proposition 2.32. Let $W \subset J^{k}(X, Y)$ be a left-right invariant submanifold. Then

1. If $f: X \rightarrow Y$ is a stable map, then $j^{k} f \pitchfork W$.
2. If $f:\left(X, x_{0}\right) \rightarrow\left(Y, y_{0}\right)$ is a map-germ of finite $\mathcal{A}$-codimension. Then $j^{k} f \pitchfork W$ on $X \backslash\left\{x_{0}\right\}$.

Proof. Suppose $f$ is stable.
Step 1: Suppose that $j^{k} f\left(x_{0}\right) \in W$. There exists a germ of unfolding $F:\left(X \times S,\left(x_{0}, 0\right)\right) \rightarrow(Y \times$ $S,\left(f\left(x_{0}\right), 0\right)$ ) of $f$ such that the "relative" jet extension map $j_{x}^{k} F: X \times S \rightarrow J^{k}(X, Y)$ is transverse to $W$ at $\left(x_{0}, 0\right)$. This can be arranged by choosing coordinates on $X$ and $Y$ around $x_{0}$ and $y_{0}$, and then taking as parameter space $S=J^{k}(n, p)$, and regarding its members as polynomial maps, which can be added to $f$. The resulting family is defined by $F(x, u)=f(x)+u(x)$, and $\left.j_{x}^{k} F\right|_{\left\{x_{0}\right\} \times S} \rightarrow J^{k}(X, Y)_{\left(x_{0}, y_{0}\right)}$ is the identity map. It is thus transverse to $W$.

Step 2: $f$ is stable, so $F$ is a trivial unfolding. Thus, there exist germs of diffeomorphisms $\Phi$ of $\left(X \times S,\left(x_{0}, 0\right)\right)$ with $\Phi(x, u)=\left(\varphi_{u}(x), u\right)$ and $\Psi$ of $\left(Y \times S,\left(y_{0}, 0\right)\right)$ with $\left.\Psi(y, u)=\psi_{u}(y), u\right)$ such that $\Psi \circ\left(f \times \operatorname{id}_{S}\right) \circ \Phi=F$. As $j_{x}^{k} F \pitchfork W$, we have $j_{x}^{k} \Psi \circ F \circ \Phi \pitchfork W$. As $W$ is left-right invariant, it follows that $j^{k} f \pitchfork W$ (Exercise).

The second statement follows by the geometric criterion for finite codimension.
Using an auxiliary map such as $j^{k} f$ to pull back a universal object from jet space can give useful information. Provided the codimension of the pulled back object is the same as the codimension of the universal object, much of the associated algebraic structure pulls back also. For example,

Theorem 2.33. Suppose $X$ and $Y$ are smooth and $W \subset Y$ is a Cohen-Macaulay space. Let $f: X \rightarrow Y$ be an analytic map. Then

$$
\begin{equation*}
\text { the codimension of } f^{-1}(W) \text { in } X \leq \text { the codimension of } W \text { in } Y \tag{2.10}
\end{equation*}
$$

and if this inequality is an equality then

1. $f^{-1}(W)$ is Cohen-Macaulay.
2. If $\mathbf{L}_{\bullet}$ is a free resolution of the germ of $\mathcal{O}_{W, w_{0}}$ as $\mathcal{O}_{Y, w_{0}}$-module, then for each $x \in f^{-1}(W)$ with $f(x)=w_{0}, \mathbf{L} \bullet \otimes_{\mathcal{O}_{Y, w_{0}}} \mathcal{O}_{X, x}$ is a free resolution of $\mathcal{O}_{f^{-1}(W), x}$ as $\mathcal{O}_{X, x}$ module.

This is proved in Subsection 2.6.
A second important application of jet-space is through the Thom Transversality Theorem, which concerns the behaviour of smooth maps between smooth manifolds. A residual subset of a topological space is the intersection of a countable number of dense open sets, and a property is generic if it is held by all members of a residual subset.

Exercise 2.34. If $A$ is a residual subset of $S$, can $S \backslash A$ contain a residual subset of $S$ ?
If $M$ and $N$ are smooth manifolds, the Whitney $C^{k}$ Topology on the space $C^{\infty}(M, N)$ of smooth maps from $M$ to $N$ has as base the collection of subsets modelled on open sets $U \subset J^{k}(M, N)$ :

$$
C_{U}=\left\{f \in C^{\infty}(M, N): j^{k} f(M) \subset U\right\}
$$

and the Whitney $C^{\infty}$ topology allows such sets for all values of $k$. It is a Baire Space - residual sets are dense. A property of mappings $M \rightarrow N$ is said to be generic if it is held by the members of a residual subset of $C^{\infty}(M, N)$.

Theorem 2.35. Let $M$ and $N$ be $C^{\infty}$ manifolds, and let $W \subset J^{k}(M, N)$ be a smooth submanifold. Then the set of smooth maps $f: M \rightarrow N$ such that $j^{k} f \pitchfork W$ is residual in $C^{\infty}(M, N)$ with the Whitney topology.

For example:
Exercise 2.36. Let $M$ be an $n$-dimensional smooth manifold and $N=\mathbb{R}^{p}$. If $p \geq 2 n$ then immersions $M \rightarrow \mathbb{R}^{p}$ are dense in $C^{\infty}\left(M, \mathbb{R}^{p}\right)$.

If codim $W>\operatorname{dim} M$, the only way that $j^{k} f: M \rightarrow J^{k}(M, N)$ can be transverse to $W$ is if $\left(j^{k} f\right)^{-1}(W) \emptyset$.

Exercise 2.37. 1. If $n<6$, the set of mappings $M^{n} \rightarrow N^{n+1}$ for which all singularities have corank 1 is residual (see 2.4 for the definition of corank).
2. What is the smallest value of $m+n$ for which a stable map $M^{m} \rightarrow N^{n}$ can have a corank 2 singularity? A corank 3 singularity?

In fact Whitney's 'easy" embedding theorem is just a short step from 2.36 :
Theorem 2.38. Let $M$ be an n-dimensional smooth manifold and $N=\mathbb{R}^{p}$. If $p \geq 2 n+1$ then the set of embeddings $M \rightarrow \mathbb{R}^{p}$ is residual in $C^{\infty}\left(M, \mathbb{R}^{p}\right)$.

An immersion is an embedding if it is a homeomorphism onto its image.
Exercise 2.39. 1. If $M$ is compact then an injective immersion is a homeomorphism onto its image.
2. An embedding is a diffeomorphism onto its image.

Properness (that the preimage of every compact set is compact) is a global property with some subtlety, and we will not discuss it except to say that it is automatic if the domain is compact. Injectivity, on the other hand, is a property of jets, and can be arranged, if the dimensions are right, by requiring transversality to a suitable submanifold of the multi-jet space ${ }_{r} J^{k}(M, N)$, which
is defined as follows: there is a natural map $p: J^{k}(M, N) \rightarrow M$ giving the source of each jet; ${ }_{r} J^{k}(M, N)$ is the preimage in $\left(J^{k}(M, N)\right)^{r}$ of the set

$$
M^{(r)}=\left\{\left(x_{1}, \ldots, x_{r}\right) \in M^{r}: x_{i} \neq x_{j} \text { if } i \neq j\right\}
$$

under the $r$-fold Cartesian product map $p^{r}:\left(J^{k}(M, N)\right)^{r} \rightarrow M^{r}$. Each map $f: M \rightarrow N$ gives rise to a natural map ${ }_{r} j^{k} f: M^{(r)} \rightarrow{ }_{r} J^{k}(M, N)$.
Theorem 2.40. Let $M$ and $N$ be $C^{\infty}$ manifolds, and let $W \subset_{r} J^{k}(M, N)$ be a smooth submanifold. Then the set of smooth maps $f: M \rightarrow N$ such that ${ }_{r} j^{k} f \pitchfork W$ is residual in $C^{\infty}(M, N)$ with the Whitney topology.

Exercise 2.41. 1. Let $M$ be compact. If $\operatorname{dim} N>2 \operatorname{dim} N$ then the set of embeddings is residual in $C^{\infty}(M, N)$.
2. A critical point $x_{0}$ of a smooth function $f: M^{m} \rightarrow \mathbb{R}$ is non-degenerate if the Hessian matrix $\left.\operatorname{det}\left(\left[\frac{\partial^{2} f}{\partial x_{i} x_{j}}\left(x_{0}\right)\right]_{1 \leq i, j \leq m}\right]\right)$ (with respect to some, and hence any, set of local coordinate) is invertible. A function $M \rightarrow \mathbb{R}$ is a Morse function if all of its critical points are nondegenerate and no two critical points share the same critical value. Show that for any smooth manifold $M$, Morse functions form a residual set in $C^{\infty}(M, \mathbb{R})$.
3. A fixed point $x_{0}$ of a smooth map $f: M \rightarrow M$ is non-degenerate if $d_{x_{0}} f$ does not have 1 as an eigenvalue. Show that this condition can be expressed in terms of the transversality of some jet extension map to a suitable submanifold of jet space, and deduce that the set of maps $f: M \rightarrow M$ with only non-degenerate fixed points is residual in $C^{\infty}(M, M)$.

Whitney's embedding theorem does not require the hypothesis of compactness, but explaining this would lead us too far away from the main thrust of the lectures.

Further reading: Chapter II of the textbook [15] of Guillemin and Golubitsky.

### 2.6 Pulling back algebraic structures

The following three results fit well with the idea that in singularity theory we study ideal objects, in the sense of Plato, and then attempt to wrestle their properties back to the reality of our concrete examples by some kind of pull-back procedure. The ideal objects are usually contained in spaces of $p \times q$ matrices, or jet spaces $J^{k}(N, P)$. The condition for the success of this strategy is usually that the codimension of the concrete object in its ambient space is the same as the codimension of the ideal object in its ambient space. As a typical example of this, let us cite two theorems, one describing ideal (generic) objects, the other describing the kinds of objects we encounter in studying singularities. Let $X=\operatorname{Mat}_{p \times q}(\mathbb{C})$ be the space of $p \times q$ matrices with complex entries. As (global) coordinates on $X$ we take the entries $x_{i j}, 1 \leq i \leq p, 1 \leq j \leq q$. We refer to the matrix of coordinates $\left(x_{i j}\right)$ as the generic matrix. Let $\mathcal{O}_{X}$ be the sheaf of germs of analytic functions on $X$, and let $I_{k}$ be the sheaf of ideals of $\mathcal{O}_{X}$ generated by the $k \times k$ minors of the generic matrix.

Theorem 2.42. ([]) $V\left(I_{k}\right)$, with structure sheaf $\mathcal{O}_{X} / I_{k}$, is Cohen Macaulay and of codimension $(p-k+1)(q-k+1)$ in $X$.

Theorem 2.43. Let $M$ be an $p \times q$ matrix with entries in $\mathcal{O}_{\mathbb{C}^{n}, 0}$ and let $\min _{k}(M)$ be the ideal in $\mathcal{O}_{\mathbb{C}^{n}}$ generated by the $k \times k$ minors of $M$, with $k \leq p, q$. Then
1.

$$
\begin{equation*}
\text { codimension of } W_{k}:=V\left(\min _{k}(M)\right) \text { in } X \leq(p-k+1)(q-k+1) \tag{2.11}
\end{equation*}
$$

2. If equality is attained in (2.11) then $Y$, with structure sheaf $\mathcal{O}_{\mathbb{C}^{n}} / \min _{k}(M)$, is Cohen Macaulay.

We will prove 2.11 from 2.43 by the "standard strategy" shortly. This strategy is described in the following sequence of results.
Lemma 2.44. Let $M$ be a Cohen-Macaulay module over the ring $R$ and let $a_{1}, \ldots, a_{m} \in R$. If $\operatorname{dim} M /\left(a_{1}, \ldots, a_{m}\right)=\operatorname{dim} M-m$ then $a_{1}, \ldots, a_{m}$ is an $M$-sequence.
Proof. We prove this by induction on $m$. Let $M_{j}=M /\left(a_{1}, \ldots, a_{j}\right) M=M_{j-1} / a_{j} M_{j-1}$. The hypothesis implies that $\operatorname{dim} M_{j} / a_{j+1} M_{j}=\operatorname{dim} M_{j}-1$. We claim that $a_{j+1}$ cannot be a member of any associated prime of $M_{j}$. For $\operatorname{Ass}\left(M_{j}\right)$ is the set of minimal members (with respect to inclusion) of $\operatorname{Supp}\left(M_{j}\right)$. The fact that $M_{j}$ is Cohen-Macaulay means in particular that all of these have the same height, equal to $\operatorname{dim} R-\operatorname{dim} M_{j}$. Because $\operatorname{dim} M_{j} / a_{j+1} M_{j}<\operatorname{dim} M_{j}$, the minimal members of $\operatorname{Ass}\left(M_{j} / a_{j+1} M_{j}\right)=\operatorname{Supp}\left(M_{j}\right) \cap V\left(a_{j+1}\right)$ are all of greater height than the minimal members of $\operatorname{supp}\left(M_{j}\right)$. Thus

$$
\text { miminal members of } \operatorname{Supp}\left(M_{j}\right) \bigcap V\left(a_{j+1}\right)
$$

contains none of the minimal members of $\operatorname{Supp}\left(M_{j}\right)$. In other words, $a_{j+1}$ lies in none of the minimal members of $\operatorname{Supp}\left(M_{j}\right)$, i.e. in none of the associated primes of $M_{j}$. This means that $a_{j+1}$ is regular on $M_{j}$.

Lemma 2.45. Suppose that $M$ is a Cohen-Macaulay module over $R$ and that the elements $a_{1}, \ldots, a_{m}$ in $R$ form an $M$-sequence. Let I be the ideal in $R$ generated by $a_{1}, \ldots, a_{m}$. If $\mathbf{L}_{\bullet}$ is a free resolution of $M$ over $R$, then $\mathbf{L} \bullet \otimes R / I$ is a free resolution of $M / I M$.
Proof. Again we use induction on $m$, and the sequence $M_{j}, j=0, \ldots, m$ of modules defined in the previous proof. Let $R_{0}=R$ and $R_{j}=R /\left(a_{1}, \ldots, a_{j}\right)$ for $j=1, \ldots, m$. Suppose that $\mathbf{L} \otimes_{\bullet} \otimes_{j}$ is exact. Then it is a resolution of $M_{j}$. We have

$$
H_{i}\left(\mathbf{L}, \otimes R_{j+1}\right)=\operatorname{Tor}_{i}^{R_{j}}\left(M_{j}, R_{j} /\left(a_{j+1}\right)\right)
$$

so to prove exactness we have to show that these Tor modules vanish. We calculate $\operatorname{Tor}^{R_{j}}\left(M_{j}, R_{j} /\left(a_{j+1}\right)\right)$ by tensoring the short exact sequence

$$
0 \longrightarrow R_{j} \xrightarrow{a_{j+1}} R_{j} \longrightarrow R_{j+1} \longrightarrow 0
$$

with $M_{j}$. This gives the long exact sequence

$$
\begin{aligned}
\cdots \rightarrow \operatorname{Tor}_{i+1}\left(M_{j}, R_{j}\right) & \rightarrow \operatorname{Tor}_{i}\left(M_{j}, R_{j+1}\right) \\
\cdots & \operatorname{Tor}_{i}\left(M_{j}, R_{j}\right) \rightarrow \cdots \\
\cdots & \operatorname{Tor}_{1}\left(M_{j}, R_{j+1}\right) \longrightarrow M_{j} \xrightarrow{a_{j+1}} M_{j} \longrightarrow M_{j+1} \longrightarrow 0
\end{aligned}
$$

From this it we immediately obtain the vanishing of $\operatorname{Tor}_{i}^{R_{j}}\left(M_{j}, R_{j+1}\right)=0$ for $i>1$, since this module appears in the sequence flanked by Tor modules which are trivially zero. Vanishing of $\operatorname{Tor}_{1}^{R_{j}}\left(M_{j}, R_{j+1}\right)$ follows from the vanishing of $\operatorname{Tor}_{1}^{R_{j}}\left(M_{j}, R_{j}\right)$ and the injectivity of $M_{j} \xrightarrow{a_{j+1}} M_{j}$.

Theorem 2.46. Suppose $X$ and $Y$ are smooth and $W \subset Y$ is a Cohen-Macaulay space. Let $f: X \rightarrow Y$ be an analytic map. Then

$$
\begin{equation*}
\text { codimension of } f^{-1}(W) \text { in } X \leq \text { codimension of } W \text { in } Y \tag{2.12}
\end{equation*}
$$

and if this inequality is an equality then

1. $f^{-1}(W)$ is Cohen-Macaulay.
2. If $\mathbf{L}$. is a free resolution of the germ of $\mathcal{O}_{W, w_{0}}$ as $\mathcal{O}_{Y, w_{0}}$-module, then for each $x \in f^{-1}(W)$ with $f(x)=w_{0}, \mathbf{L}_{\bullet} \otimes_{\mathcal{O}_{Y, w_{0}}} \mathcal{O}_{X, x}$ is a free resolution of $\mathcal{O}_{f^{-1}(W), x}$ as $\mathcal{O}_{X, x}$ module.
Proof. The map

$$
f^{-1}(W) \xrightarrow{\operatorname{graph} f} \widetilde{f^{-1}(W)}:=\{(x, w) \in X \times W: w=f(x)\}
$$

defined by $\operatorname{graph}(f)(x)=(x, f(x))$ has inverse given by the restriction to $\widetilde{f^{-1}(W)}$ of the usual projection $X \times W \omega X$. Thus $f^{-1}(W)$ and $\widehat{f^{-1}(W)}$ are isomorphic, and it is enough to prove that $\widehat{f^{-1}(W)}$ is Cohen Macaulay. As the product of a smooth space with a Cohen Macaulay space, $X \times W$ is Cohen Macaulay of dimension $\operatorname{dim} W+\operatorname{dim} X$. Taking local coordinates $y_{1}, \ldots, y_{p}$ on $Y$ around $w_{0}$, we can then view $\widetilde{f^{-1}(W)}$ as the fibre over $0 \in \mathbb{C}^{p}$ of the map $\pi: X \times W \rightarrow Y$ given by $\pi(x)=\left(y_{1}-f_{1}(x), \ldots, y_{p}-f_{p}(x)\right)$. By the hauptidealsatz, $\operatorname{dim} X \times W-\operatorname{dim} \underset{f^{-1}(W) \leq p=\operatorname{dim} Y, ~}{\text { }}$. from which (2.12) follows.

Now suppose that (2.12) is an equality. Then by Lemma 2.44 the components of $\pi$ form a regular sequence in $\mathcal{O}_{X \times W}$. Since $\mathcal{O}_{X \times W}$ is Cohen-Macaulay, so is $\mathcal{O}_{X \times W} /\left(y_{1}-f_{1}(x), \ldots, y_{p}-f_{p}(x)\right)=$ $\mathcal{O} \widetilde{f^{-1}(W)} \simeq \mathcal{O}_{f^{-1}(W)}$. This proves that $f^{-1}(W)$ is Cohen-Macaulay. The remaining statement is just Lemma 2.45 applied to the $\mathcal{O}_{X \times W^{\prime}}$-module $\mathcal{O} \widetilde{f^{-1}(W)}$.

As a sample of the application of (part of) this result, we give the following
Proof of 2.11 from 2.43. Denote the entries of $M$ by $m_{i j}$. Let $\psi_{M}$ denote the map

$$
\mathbb{C}^{n} \rightarrow \operatorname{Mat}_{p \times q}(\mathbb{C}), \quad x \mapsto\left(m_{i j}(x): 1 \leq i \leq p, 1 \leq j \leq q\right)
$$

Then $V\left(\min _{k}(M)\right)=\psi_{M}^{-1}\left(W_{k}\right)$. Now apply Theorem 2.46 .
Corollary 2.47. Let $f:\left(\mathbb{C}^{n}, 0\right) \rightarrow\left(\mathbb{C}^{p}, 0\right)$ be an analytic germ, with $n<p$, and denote by $\sum_{f}$ the non-immersive locus of $f$. Then

$$
\operatorname{codim}\left(\sum_{f}\right) \leq p-n+1
$$

and in case of equality, $\sum_{f}$ is Cohen Macaulay.
Proof. $\sum_{f}$ is defined by the maximal $(=n \times n)$ minors of the Jacobian matrix

$$
\left(\frac{\partial f_{i}}{\partial x_{j}} 1 \leq i \leq p, 1 \leq j \leq n\right)
$$

of $f$. So the corollary is just an application of 2.11 .

## 3 Left-right equivalence of germs of mappings

Let $f, g:(X, S) \rightarrow\left(Y, y_{0}\right)$ be germs of analytic maps. They are

1. right-equivalent if there exists a germ of bianalytic map $\varphi:(X, S) \rightarrow(X, S)$ such that $f_{2}=$ $f_{1} \circ \varphi$;
2. left equivalent, if there exists a germ of bianalytic map $\psi:\left(Y, y_{0}\right) \rightarrow\left(Y, y_{0}\right)$ such that $f_{2}=$ $\psi \circ f_{1} ;$
3. left-right-equivalent, if there exist germs of bianalytic maps $\varphi:(X, S) \rightarrow(X, S)$ and $\psi$ : $\left(Y, y_{0}\right) \rightarrow\left(Y, y_{0}\right)$ such that $\psi \circ f \circ \varphi^{-1}=g$. This is the most natural equivalence relation if one is interested in the maps themselves.
4. contact equivalent, if there exists a germ of diffeomeorphism $\Phi:\left(X \times Y, S \times\left\{y_{0}\right\}\right) \rightarrow(X \times$ $\left.Y, S \times\left\{y_{0}\right\}\right)$, of the form $\Phi(x, y)=\left(\varphi_{1}(x), \varphi_{2}(x, y)\right)$, such that $\Phi\left(\operatorname{graph}\left(f_{1}\right)\right)=\operatorname{graph}\left(f_{2}\right)$.

The term "bianalytic map" is usually replaced by "diffeomorphism", perhaps because of the fact that a great deal of the theory works unchanged for $C^{\infty}$ maps. In each case there is a group of germs of diffeomorphisms acting on the set of mappings. The groups (or, more precisely, their actions) are denoted by $\mathcal{R}, \mathcal{L}, \mathcal{A}$ and $\mathcal{K}$ respectively. We will be most interested in $\mathcal{A}$ : it is the most natural if one is interested in the geometry of maps between complex spaces.

Exercise 3.1. 1. Show that $\mathcal{R} \subset \mathcal{K}$, in the sense that $\mathcal{R}$-equivalence implies $\mathcal{K}$-equivalence.
2. Show that if $f \sim_{\mathcal{K}} g$ then $f^{-1}\left(y_{0}\right)$ and $g^{-1}\left(y_{0}\right)$ are diffeomorphic.

For a very good survey of these groups and their actions, see [49].
A big part of singularity theory has always been concerned with the problem of classification. Generally one classifies germs of analytic maps $\left(\mathbb{C}^{n}, S\right) \rightarrow\left(\mathbb{C}^{p}, 0\right)$ up to $\mathcal{A}$-equivalence, and up to $\mathcal{R}$-equivalence if $p=1$. Contact equivalence is a technical device which is of interest primarily if one is concerned with preimages of $y_{0}$, but also plays a role in the theory of left-right equivalence, as we will see.

A key ingredient in classification is the notion of finite determinacy. Let us assume that $X=\mathbb{C}^{n}$, $Y=\mathbb{C}^{p}$ and $S=\left\{x_{0}\right\}$.

Definition 3.2. Let $f:\left(X, x_{0}\right) \rightarrow\left(Y, y_{0}\right)$ be a complex analytic or $\mathcal{C}^{\infty}$ map, and let $\mathcal{G}$ be one of the groups listed above. We say $f$ is $k$-determined for $\mathcal{G}$-equivalence if whenever the Taylor series of another germ $g$ coincides with that of $f$ up to degree $k$, then $f \sim_{\mathcal{G}} g$, and finitely determined if it is $k$-determined for some $k \in \mathbb{N}$.

The notion has an obvious generalisation to the case where $S$ consists of more than a single point, but has only been used in practice in case $S$ is a finite point set. Here we will look only at the case where $S$ is a single point.

In [29], John Mather showed that for all of the groups listed above, finite determinacy is equivalent to isolated instability. We will not prove this, but will explain the main ideas of the proof. The key is to understand how to construct diffeomorphisms. In all of singularity theory this is done by integrating vector fields. With very few exceptions, there is no other method!

### 3.1 Integration of vector fields

Proposition 3.3. Let $\chi$ be an analytic vector field on the open set $U \subset \mathbb{C}^{n}$. Then for each $x_{0} \in U$ there is an open neighbourhood $U\left(x_{0}\right) \subset U$, a disc $B_{\eta}(0)$ of radius $\eta>0$ centred at $0 \in \mathbb{C}$, and an analytic map $\Phi: U\left(x_{0}\right) \times B_{\eta}(0) \rightarrow U$ such that for all $(x, t)$

1. $\Phi(x, 0)=x$;
2. $\frac{d}{d t} \Phi(x, t)=\chi(\Phi(x, t))$.

The curve described by $\Phi(x, t)$, for fixed $x$, as $t$ varies, is called a trajectory of the vector field $\chi$, and (2) above says that the tangent vector to this trajectory at the point $\Phi(x, t)$ is the vector $\chi(\Phi(x, t))$. Writing $\gamma_{x}(t)$ in place of $\Phi(x, t)$, and keeping $x$ fixed, this becomes

$$
\gamma_{x}^{\prime}(t)=\chi\left(\gamma_{x}(t)\right) .
$$

If instead we fix $t$, we get a map $\varphi_{t}: U\left(x_{0}\right) \rightarrow U$. Notice that (1) above says that $\varphi_{0}$ is the identity map. From the theorem of existence and uniqueness of solutions of ordinary differential equations, one easily deduces

Corollary 3.4. 1. Wherever the composite is defined, one has

$$
\varphi_{s} \circ \varphi_{t}=\varphi_{s+t} .
$$

2. For each $x_{0} \in U$ and each fixed value of $t \in B_{\eta}(0)$, the map $\varphi_{t}: U\left(x_{0}\right) \rightarrow \varphi_{t}\left(U\left(x_{0}\right)\right)$ is a diffeomorphism (bianalytic isomorphism), with inverse $\varphi_{-t}$.

The family of diffeomorphisms $\varphi_{t}$ is called the integral flow of the vector field $\chi$. All arguments involving the integration of vector fields to construct diffeomorphisms go via the following ThomLevine theorem:

Corollary 3.5. Suppose that $F: X \rightarrow Y$ is an analytic map of complex manifolds, and that $\chi$ and $\tilde{\chi}$ are vector fields on $Y$ and $X$ such that for each $x \in X$ one has

$$
\begin{equation*}
d_{x} F(\tilde{\chi}(x))=\chi(F(x)) . \tag{3.1}
\end{equation*}
$$

Then the integral flows $\Phi$ and $\tilde{\Phi}$ of $\chi$ and $\tilde{\chi}$ satisfy

$$
\begin{equation*}
F \circ \tilde{\varphi}_{t}=\varphi_{t} \circ F \tag{3.2}
\end{equation*}
$$

wherever the composites are defined.
The two equations (3.1) and (3.2) can be expressed in terms of commutative diagrams. The vector fields $\chi$ and $\tilde{\chi}$ are sections of the tangent bundles $T Y$ and $T X$ respectively, and (3.1) and (3.2) say that the diagrams

commute.
The Thom-Levine theorem shows how an "infinitesimal condition" gives rise to a family of diffeomorphisms. Equalities like (3.1) are linear in $\chi$ and $\tilde{\chi}$, and these vector fields can often be constructed by the methods of commutative algebra. This is the entry-point of commutative algebra, which, through it, has a huge input into Singularity Theory.

As an example of what is involved, let us prove the simplest of the determinacy theorems of John Mather. If $f:\left(\mathbb{C}^{n}, 0\right) \rightarrow \mathbb{C}$ is an analytic germ of function, then the first order partial derivatives $\partial f / \partial x_{1}, \ldots, \partial f / \partial x_{n}$ generate the jacobian ideal $J_{f}$ in the ring $\mathcal{O}_{\mathbb{C}^{n}, 0}$.

Example 3.6. 1. If $f\left(x_{1}, \ldots, x_{n}\right)=x_{1}^{2}+\cdots+x_{n}^{2}$ then $J_{f}$ is the maximal ideal $\mathfrak{m}:=\mathfrak{m}_{\mathbb{C}^{n}, 0}$.
2. If $f\left(x_{1}, x_{2}\right)=x_{1}^{2}+x_{2}^{k+1}$ then $J_{f}=\left(x_{1}, x_{2}^{k}\right)$.
3. If $f\left(x_{1}, x_{2}\right)=x_{1}^{2} x-2+x_{2}^{k-1}$ then $J_{f}=\left(x_{1} x_{2}, x_{1}^{2}+(k-1) x_{2}^{k-2}\right)$.
4. If $f\left(x_{1}, x_{2}\right)=x_{1}^{2} x_{2}$ then $J_{f}=\left(x_{1} x_{2}, x_{1}^{2}\right)$.

Theorem 3.7. (i) Suppose that $f \in \mathcal{O}_{\mathbb{C}^{n}, 0}$ is $k$-determined for right equivalence. Then $\mathfrak{m} J_{f} \supset$ $\mathfrak{m}^{k+1}$.
(ii) Conversely, suppose that $f \in \mathcal{O}_{\mathbb{C}^{n}, 0}$ and

$$
\begin{equation*}
\mathfrak{m} J_{f} \supset \mathfrak{m}^{k} \tag{3.4}
\end{equation*}
$$

Then $f$ is $k$-determined for $\mathcal{R}$-equivalence.
Exercise 3.8. Find the lowest value of $k$ for which (3.4) holds for each of the functions in Example 3.6 .

Proof of 3.7.(i) Let $h \in \mathfrak{m}^{k+1}$. Then for all $t$ there exists $\varphi_{t} \in \operatorname{Diff}\left(\mathbb{C}^{n}, 0\right)$ such that $f+t h=f \circ \varphi_{t}$. If we could assume the existence of a smoothly parametrised family of diffeomorphisms $\varphi_{t}$ with $\varphi_{0}=$ id such that $f \circ \varphi_{t}=f+t h$ then we could reason as follows:

$$
\begin{equation*}
h=\frac{\partial(f+t h)}{d t}=\frac{\partial\left(f \circ \varphi_{t}\right)}{d t}=\sum_{i}\left(\frac{\partial f}{\partial x_{i}} \circ \varphi_{t}\right) \frac{\partial \varphi_{t, i}}{\partial t} . \tag{3.5}
\end{equation*}
$$

Note that since $\varphi_{t}(0)=0$ for all $t$ it follows that $\partial \varphi_{t, i} / \partial t \in \mathfrak{m}$. When $t=0$, since $\varphi_{0}=\mathrm{id}$, this gives

$$
\begin{equation*}
h=\sum_{i} \frac{\partial f}{\partial x_{i}} \frac{\partial \varphi_{t, i}}{\partial t} \in \mathfrak{m} J_{f} \tag{3.6}
\end{equation*}
$$

so that $\mathfrak{m}^{k+1} \subset \mathfrak{m} J_{f}$ as required.
However, our hypothesis does not allow us immediately to assert that the diffeomorphisms $\varphi_{t}$ fit together to give a smooth family. So instead we look in jet space $J^{k+1}(n, 1)=\mathfrak{m}_{n} / \mathfrak{m}_{n}^{k+2}$. As $f$ is $k$-determined, the set

$$
L:=\left\{j^{k+1}(f+h): h \in \mathfrak{m}^{k+1}\right\} \subset J^{k+1}(n, 1)
$$

lies entirely in the $\mathcal{R}^{(k+1)}$-orbit of $f$, where $\mathcal{R}^{(k+1)}$ is the finite dimensional quotient of $\operatorname{Diff}\left(\mathbb{C}^{n}, 0\right)$ acting on jet space. Now $\mathcal{R}^{(k+1)}$ can be identified with the set

$$
\left\{j^{k+1} \varphi(0): \varphi \in \operatorname{Diff}\left(\mathbb{C}^{n}, 0\right)\right\}
$$

and has a natural structure of algebraic group: the composite of two polynomial mappings depends polynomially on their coefficients, and in $\mathcal{R}^{(k+1)}$ one composes and then truncates at degree $k+1$. This group acts algebraically on $J^{k+1}(n, 1)$. Thus, as the set $L$ lies in the orbit of $j^{k+1} f(0)$, writing $z=j^{k+1} f(0)$, and $\mathcal{R}^{(k+1)} z$ for the $\mathcal{R}^{(k+1)}$-orbit of $z$, one has

$$
\begin{equation*}
\frac{\mathfrak{m}^{k+1}}{\mathfrak{m}^{k+2}}=T_{z} L \subset T_{z}\left(\mathcal{R}^{(k+1)} z\right)=\frac{\mathfrak{m} J_{f}+\mathfrak{m}^{k+2}}{\mathfrak{m}^{k+2}} \tag{3.7}
\end{equation*}
$$

and thus

$$
\mathfrak{m}^{k+1} \subset \mathfrak{m} J_{f}+\mathfrak{m}^{k+2}
$$

The conclusion we want follows by Nakayama's Lemma, 2.5.
The second equality in (3.7) is important and not completely obvious. It can be obtained along the lines of the argument leading up to (3.6), but using the crucial fact that if the Lie group $G$ acts on the manifold $M$ and for $x \in M$ we denote by $\alpha_{x}$ the orbit map $g \in G \mapsto g x$, then for each $x \in M$ with smooth orbit $G x$,

$$
T_{x} G x=d_{e} \alpha_{x}\left(T_{e} G\right)
$$

Now

$$
d_{e} \alpha_{x}\left(T_{e} G\right)=\left\{\left.\frac{d}{d t}(\gamma(t) \cdot x)\right|_{t=0}: \gamma:(\mathbb{C}, 0) \rightarrow(G, e) \text { is a curve germ }\right\}
$$

every curve in $\left(\mathcal{R}^{(k+1)}, \mathrm{id}\right)$ is of the form $j^{k+1} \varphi_{t}$ for a 1-parameter family of diffeomorphisms $\varphi_{t}$, so now it really is true that

$$
\begin{aligned}
T_{z} \mathcal{R}^{(k+1)} & =\left\{\frac{d}{d t} j^{k+1}\left(f \circ \varphi_{t}\right)_{t=0}: \varphi_{t} \text { is a 1-parameter family in Diff }\left(\mathbb{C}^{n}, 0\right) \text { with } \varphi_{0}=\mathrm{id}\right\} \\
& =\left\{j^{k+1}\left(\left.\frac{d}{d t}\left(f \circ \varphi_{t}\right)\right|_{t=0}\right): \varphi_{t} \text { is a 1-parameter family in } \operatorname{Diff}\left(\mathbb{C}^{n}, 0\right) \text { with } \varphi_{0}=\mathrm{id}\right\}
\end{aligned}
$$

(ii) Suppose that $g$ has the same degree $k$ Taylor polynomial as $f$. Then $g-f \in \mathfrak{m}^{k+1}$. Let $F(x, t)=f(x)+t(g(x)-f(x))$, and write $f_{t}(x)=F(t, x)$. The idea of the proof is to show that for each value $t_{0}$ of $t$, there is a neighbourhood $U\left(t_{0}\right)$ of $t_{0}$ in $\mathbb{C}$ such that $f_{t}$ and $f_{t_{0}}$ are $\mathcal{R}$-equivalent for all $t \in U\left(t_{0}\right)$. A finite number of these neighbourhoods cover the compact interval $[0,1] \subset \mathbb{C}$, so by transitivity $f=f_{0} \sim_{\mathcal{R}} f_{1}=g$.

We do this first for $t_{0}=0$. As $F$ is a function of the $n+1$ variables $x_{1}, \ldots, x_{n}$, $t$, we consider the germ $F \in \mathcal{O}_{\mathbb{C}^{n+1}, 0}$. Notice that $\partial F / \partial t=g-f \in \mathfrak{m}_{n}^{k+1}$, where by $\mathfrak{m}_{n}$ we mean the ideal in $\mathcal{O}_{\mathbb{C}^{n+1}, 0}$ generated by $\left(x_{1}, \ldots, x_{n}\right)$. This is of course not the maximal ideal of $\mathcal{O}_{\mathbb{C}^{n+1}, 0}$. In any case, it follows from our hypothesis on $f$ that

$$
\begin{equation*}
\frac{\partial F}{\partial t} \in \mathfrak{m}_{n}\left(\frac{\partial f}{\partial x_{1}}, \ldots, \frac{\partial f}{\partial x_{n}}\right) . \tag{3.8}
\end{equation*}
$$

We would like to show

$$
\begin{equation*}
\frac{\partial F}{\partial t} \in \mathfrak{m}_{n}\left(\frac{\partial F}{\partial x_{1}}, \ldots, \frac{\partial F}{\partial x_{n}}\right) . \tag{3.9}
\end{equation*}
$$

For if we have

$$
\begin{equation*}
\frac{\partial F}{\partial t}=\tilde{\chi}_{1} \frac{\partial F}{\partial x_{1}}+\cdots+\tilde{\chi}_{n} \frac{\partial F}{\partial x_{n}} \tag{3.10}
\end{equation*}
$$

for some functions $\tilde{\chi}_{i} \in \mathfrak{m} \mathcal{O}_{\mathbb{C}^{n+1}, 0}$, then defining a vector field $\tilde{\chi}$ on $\mathbb{C}^{n+1}$ by

$$
\tilde{\chi}=\frac{\partial}{\partial t}-\sum_{i} \chi_{i} \frac{\partial}{\partial x_{i}},
$$

(3.10) becomes

$$
d F(\tilde{\chi})=0 .
$$

This is exactly (3.1) with $\chi=0$. Let $\tilde{\Phi}(x, t)=\left(\tilde{\phi}_{t}(x), t\right)$ be the integral flow of the vector field $\tilde{\chi}$. The integral flow of the zero vector field is the identity map, and therefore by the Thom-Levine lemma we have

$$
\begin{equation*}
F \circ \tilde{\Phi}=F . \tag{3.11}
\end{equation*}
$$

Since the component of $\tilde{\chi}$ in the $t$-direction has constant length 1 , it follows that $\tilde{\varphi}_{t}$ maps $\mathbb{C}^{n} \times\{0\}$ to $\mathbb{C}^{n} \times\{t\}$. Restricting both sides of (3.11) to $\mathbb{C}^{n} \times\{0\}$ we therefore get

$$
f_{t} \circ \tilde{\varphi}_{t}=f .
$$

This is not quite enough to show that the germs at 0 of $f$ and of $f_{t}$ are right-equivalent: we need to show also that $\varphi_{t}(0)=0$. But this holds, because $\tilde{\chi}_{i} \in \mathfrak{m}_{n}$ for all $i$. Thus $\tilde{\varphi}_{t} \in \mathcal{R}$ and $f_{t} \sim_{\mathcal{R}} f$ as required.


The arrows show a real version of the vector field $\tilde{\chi}$ of the proof. At all points of the $t$-axis, the vector field is tangent to the axis, so any trajectory beginning at a point on the axis remains on the axis. Thus $\varphi_{t}(0)=0$.

Now we set about deducing (3.9) from (3.8). Since $\partial F / \partial t=g-f \in \mathfrak{m}_{n}^{k}$, to show (3.9), it will be enough to show

$$
\begin{equation*}
\mathfrak{m}_{n}^{k} \subset \mathfrak{m}_{n}\left(\frac{\partial F}{\partial x_{1}}, \ldots, \frac{\partial F}{\partial x_{n}}\right) . \tag{3.12}
\end{equation*}
$$

We know that

$$
\begin{equation*}
\mathfrak{m}_{n}^{k} \subset \mathfrak{m}_{n}\left(\frac{\partial f}{\partial x_{1}}, \ldots, \frac{\partial f}{\partial x_{n}}\right) \tag{3.13}
\end{equation*}
$$

(as ideals in $\mathcal{O}_{\mathbb{C}^{n+1}, 0}$ as in $\mathcal{O}_{\mathbb{C}^{n}, 0}$ ). Because

$$
\begin{equation*}
\frac{\partial f}{\partial x_{i}}=\frac{\partial F}{\partial x_{i}}-t \frac{\partial(g-f)}{\partial x_{i}} \tag{3.14}
\end{equation*}
$$

it follows that

$$
\frac{\partial f}{\partial x_{i}} \in\left(\frac{\partial F}{\partial x_{1}}, \ldots, \frac{\partial F}{\partial x_{n}}\right)+\mathfrak{m}_{n+1} \mathfrak{m}^{k}
$$

and therefore

$$
\begin{equation*}
\mathfrak{m}_{n}^{k} \subset\left(\frac{\partial f}{\partial x_{1}}, \ldots, \frac{\partial f}{\partial x_{n}}\right) \subset\left(\frac{\partial F}{\partial x_{1}}, \ldots, \frac{\partial F}{\partial x_{n}}\right)+\mathfrak{m}_{n+1} \mathfrak{m}^{k} . \tag{3.15}
\end{equation*}
$$

Now some commutative algebra comes to our aid. By Nakayama's Lemma, 2.5, proved in Subsection 2.2 , (3.15) implies at once that (3.12) holds: we apply it taking as $R$ the local ring $\mathcal{O}_{\mathbb{C}^{n+1}}$ with maximal ideal $\mathfrak{m}_{n+1}$, and taking $M=\mathfrak{m}_{n}^{k}$ and $N=\mathfrak{m}_{n} J_{f}$ (where, as before $\mathfrak{m}_{n}$ means the ideal in $\mathcal{O}_{\mathbb{C}^{n+1}}$ generated by $\left.x_{1}, \ldots, x_{n}\right)$.

This completes the proof that the deformation $f+t(g-f)$ is trivial for $t$ in some neighbourhood of 0 . The remainder of the proof involves showing that the same procedure can be employed for every value of $t$ : we want to show that for any $t_{0}$ the deformation $f+t(g-f)$ is trivial in a neighbourhood of $t_{0}$. This deformation can be written in the form $\left(f+t_{0}(g-f)\right)+\left(t-t_{0}\right)(g-f)$, and taking as new parameter $s=t-t_{0}$, the problem reduces to what we have already discussed, except that instead of our original $f$ we now have a new function, $f_{t_{0}}:=f+t_{0}(g-f)$. In order that our earlier argument should apply, we have to show that $f_{t_{0}}$ also satisfies the hypothesis of this argument: that

$$
\begin{equation*}
\mathfrak{m} J_{f_{t_{0}}} \supset \mathfrak{m}^{k} \tag{3.16}
\end{equation*}
$$

Once again this is done by a simple argument involving Nakayama's Lemma, which I leave as an exercise.

Exercise 3.9. Show that if $\mathfrak{m} J_{f} \supset \mathfrak{m}^{k}$ and $g-f \in \mathfrak{m}^{k+1}$ then $\mathfrak{m} J_{f_{t_{0}}} \supset \mathfrak{m}^{k}$.

The first part of the proof of Theorem 3.7 justifies part (i) of the following definition.

## Definition 3.10.

$$
\begin{aligned}
& \text { (i) } T \mathcal{R} f=\mathfrak{m}_{n} J_{f} \\
& \text { (ii) } T \mathcal{R}_{e} f=J_{f}
\end{aligned}
$$

The second tangent space is the extended right tangent space. Its heuristic justification is less clear than that of $T \mathcal{R} f$; it can be obtained by the argument of the proof of Theorem 3.7(i) if we remove the requirement that $\varphi_{t}(0)=0$ for all $t$.

In these lectures we are interested in left-right equivalence more than right equivalence. But Theorem 3.7 is a good indication of what is true and how, in principle, one goes about proving it. For left-right equivalence, the proof is necessarily more complicated, since one has simultaneously
to produce families of diffeomorphisms of source and target. However the overall strategy is the same. First we need to define a suitable tangent space for $\mathcal{A}$-equivalence.

Mather and Thom, in their work in the 60 's on smooth maps, thought in global terms: a $C^{\infty}$ map $f: N \rightarrow P$ is stable if its orbit under the natural action of $\operatorname{Diff}(N) \times \operatorname{Diff}(P)$ is open in $C^{\infty}(N, P)$, with respect to a suitable topology. Here we are interested in local geometry, and so we give a local version of this definition: a map-germ $f:\left(\mathbb{C}^{n}, 0\right) \rightarrow\left(\mathbb{C}^{p}, 0\right)$ is stable if every deformation is trivial: roughly speaking, if $f_{t}$ is a deformation of $f$ then there should exist deformations of the identity maps of $\left(\mathbb{C}^{n}, 0\right)$ and $\left(\mathbb{C}^{p}, 0\right), \varphi_{t}$ and $\psi_{t}$, such that

$$
\begin{equation*}
f_{t}=\psi_{t} \circ f \circ \varphi_{t} . \tag{3.17}
\end{equation*}
$$

A substantial part of Mather's six papers on the stability of $C^{\infty}$ mappings [28]-[33] is devoted to showing that if all the germs of a mapping $f$ are stable in this local sense then $f$ is stable in the global sense. We will not discuss global stability any further.

Definition 3.11. (1) An unfolding of $f$ is a map-germ

$$
F:\left(\mathbb{C}^{n} \times \mathbb{C}^{d}, 0\right) \rightarrow\left(\mathbb{C}^{p} \times \mathbb{C}^{d}, 0\right)
$$

of the form

$$
F(x, u)=(\tilde{f}(x, u), u)
$$

such that $\tilde{f}(x, 0)=f(x)$.
Retaining the parameters $u$ in the second component of the map makes the following definition easier to write down:
(2) The unfolding $F$ is trivial if there exist germs of diffeomorphisms

$$
\Phi:\left(\mathbb{C}^{n} \times \mathbb{C}^{d}, 0\right) \rightarrow\left(\mathbb{C}^{n} \times \mathbb{C}^{d}, 0\right)
$$

and

$$
\Psi:\left(\mathbb{C}^{p} \times \mathbb{C}^{d}, 0\right) \rightarrow\left(\mathbb{C}^{p} \times \mathbb{C}^{d}, 0\right)
$$

such that

1. $\Phi(x, u)=(\varphi(x, u), u)$ and $\varphi(x, 0)=x$
2. $\Psi(y, h)=(\psi(y, u), u)$ and $\psi(y, 0)=y$
3. 

$$
\begin{equation*}
F=\Psi \circ(f \times \mathrm{id}) \circ \Phi \tag{3.18}
\end{equation*}
$$

(where $f \times$ id is the 'constant' unfolding $(x, u) \mapsto(f(x), u)$ ).
(3) The map-germ $f:\left(\mathbb{C}^{n}, 0\right) \rightarrow\left(\mathbb{C}^{p}, 0\right)$ is stable if every unfolding of $f$ is trivial.

By writing $\varphi(x, u)=\varphi_{u}(x)$ and $\psi(y, u)=\psi_{u}(y)$, from 3.18 we recover the heuristic definition 3.17. We do not insist that the mappings $\varphi_{u}$ and $\psi_{u}$ preserve the origin of $\mathbb{C}^{n}$ and $\mathbb{C}^{p}$ respectively. After all, if the interesting behaviour merely changes its location, we should not regard the unfolding as non-trivial.

Example 3.12. Consider the map-germ $f(x)=x^{2}$, and its unfolding $F(x, u)=\left(x^{2}+u x, u\right)$. This is trivialised by the families of diffeomorphisms $\Phi(x, u)=(x+u / 2, u), \Psi(y, u)=\left(y-u^{2} / 4, u\right)$. Both $\Phi$ and $\Psi$ are just families of translations.

Exercise 3.13. Check that in the previous example $F=\Psi \circ(f \times \mathrm{id}) \circ \Phi$.

Fortunately, there exists a simple and computable criterion for stability. If $f$ is stable, then the quotient

$$
\begin{equation*}
T^{1}(f):=\frac{\left\{\left.\frac{d}{d t} f_{t}\right|_{t=0}: f_{0}=f\right\}}{\left\{\left.\frac{d}{d t}\left(\psi_{t} \circ f \circ \varphi_{t}\right)\right|_{t=0}: \varphi_{0}=\operatorname{id}\right\}}, \tag{3.19}
\end{equation*}
$$

is equal to 0 . In general this quotient is a vector space whose dimension measures the failure of stability. Mather ([30]) proved
Theorem 3.14. Infinitesimal stability is equivalent to stability: $f$ is stable if and only if $T^{1}(f)=0$.
One of the aims of this lecture is to develop techniques for calculating $T^{1}(f)$, and apply them in some examples.

Exercise 3.15. Germs of submersions and immersions are infinitesimally stable and therefore stable. This is an easy calculation using the normal forms of Theorems 2.1 and 2.2

Before continuing, we note that the denominator in (3.19) is very close to being the tangent space to the orbit of $f$ under the group $\mathscr{A}=\operatorname{Diff}\left(\mathbb{C}^{n}, 0\right) \times \operatorname{Diff}\left(\mathbb{C}^{p}, 0\right)$. It is not quite equal to it, because we are allowing $\phi_{t}$ and $\psi_{t}$ to move the origin (so they are not "paths in Diff( $\left.\mathbb{C}^{n}, 0\right)$ and Diff( $\left(\mathbb{C}^{p}, 0\right)$ "). For this reason we write it as $T \mathscr{A}_{e} f$ and call it the 'extended' tangent space. The tangent space to the $\mathscr{A}$-orbit of $f$ is denoted $T \mathscr{A} f$. We have

$$
\begin{equation*}
t f\left(m_{\mathbb{C}^{n}, 0} \theta_{\mathbb{C}^{n}, 0}\right)+\omega f\left(m_{\mathbb{C}^{p}, 0} \theta_{\mathbb{C}^{p}, 0}\right) ; \tag{3.20}
\end{equation*}
$$

the maximal ideal $m_{\mathbb{C}^{n}, 0}$ appears here because since $\varphi_{t}(0)=0$ for all $t, d \varphi_{t} /\left.d t\right|_{t=0}$ vanishes at 0 and thus belongs to $m_{\mathbb{C}^{n}, 0} \theta_{\mathbb{C}^{n}, 0}$; similarly for $m_{\mathbb{C}^{p}, 0}$.

By the chain rule,

$$
\left.\frac{d}{d t}\left(\psi_{t} \circ f \circ \varphi_{t}\right)\right|_{t=0}=d f\left(\left.\frac{d \phi_{t}}{d t}\right|_{t=0}\right)+\left(\left.\frac{d \psi_{t}}{d t}\right|_{t=0}\right) \circ f .
$$

Both $\left.\left(d \varphi_{t} / d t\right)\right|_{t=0}$ and $\left.\left(d \psi_{t} / d t\right)\right|_{t=0}$ are germs of vector fields, on $\left(\mathbb{C}^{n}, 0\right)$ and $\left(\mathbb{C}^{p}, 0\right)$ respectively: quite simply, $\left.\left(d \varphi_{t}(x) / d t\right)\right|_{t=0}$ is the tangent vector at $x$ to the trajectory $\varphi_{t}(x)$. In the same way, the elements of the numerator of 3.19 should be thought of as 'vector fields along $f^{\prime}$ '; $\left.\left(d f_{t} / d t\right)\right|_{t=0}$ is the tangent vector at $f(x)$ to the trajectory $x \mapsto f_{t}(x)$. By associating to $\left.\left(d f_{t} / d t\right)\right|_{t=0}$ the map

$$
\hat{f}: x \mapsto\left(x,\left.(d / d t) f_{t}\right|_{t=0}\right) \in T \mathbb{C}^{p}
$$

we obtain a commutative diagram:

in which the vertical maps are the bundle projections. Elements of $\theta_{\mathbb{C}^{n}, 0}$ can be written in various ways: as $n$-tuples,

$$
\xi(x)=\left(\xi_{1}(x), \ldots, \xi_{n}(x)\right)
$$

(sometimes as columns rather than rows), or as sums:

$$
\xi(x)=\sum_{j=1}^{n} \xi_{j}(x) \partial / \partial x_{j} .
$$

The second notation emphasizes the role of the coordinate system on $\mathbb{C}^{n}, 0$. Similarly, elements of $\theta(f)$ can be written as row vectors or column vectors, or as sums:

$$
\hat{f}(x)=\sum_{j=1}^{p} \hat{f}_{j}(x) \partial / \partial y_{j} .
$$

We denote by

| $\theta(f)$ | the numerator of $(3.19)$ |
| :--- | :--- |
| $\theta_{\mathbb{C}^{n}, 0}$ | the space of germs at 0 of vector fields on $\mathbb{C}^{n}$ |
| $\theta_{\mathbb{C}^{p}, 0}$ | the space of germs at 0 of vector fields on $\mathbb{C}^{p}$ |
| $t f: \theta_{\mathbb{C}^{n}, 0} \rightarrow \theta(f)$ | the map $\xi \mapsto d f \circ \xi$ |
| $\omega f: \theta_{\mathbb{C}^{p}, 0} \rightarrow \theta(f)$ | the map $\eta \mapsto \eta \circ f$ |

The notation " $t f$ " is slightly fussy. We use it instead of $d f$ here because we think of $d f$ as the bundle map between tangent bundles induced by $f$, as in the diagram (3.21), whereas $t f$ is the map "left composition with $d f$ " from $\theta_{\mathbb{C}^{n}, 0}$ to $\theta(f)$. Some authors use "df" for both. In any case,

$$
\begin{equation*}
T^{1}(f)=\frac{\theta(f)}{t f\left(\theta_{\mathbb{C}^{n}, 0}\right)+\omega f\left(\theta_{\mathbb{C}^{p}, 0}\right)}=: \frac{\theta(f)}{T \mathscr{A}_{e} f} . \tag{3.22}
\end{equation*}
$$

These spaces are not just vector spaces:

| $\theta_{\mathbb{C}^{n}, 0}$ | is an $\mathcal{O}_{\mathbb{C}^{n}, 0}$-module |
| :--- | :--- |
| $\theta(f)$ | is an $\mathcal{O}_{\mathbb{C}^{n}, 0}$-module |
| $t f: \theta_{\mathbb{C}^{n}, 0} \rightarrow \theta(f)$ | is $\mathcal{O}_{\mathbb{C}^{n}, 0}$-linear, so |
| $\theta(f) / t f\left(\theta_{\mathbb{C}^{n}, 0}\right)$ | is an $\mathcal{O}_{\mathbb{C}^{n}, 0}$-module |

But $T^{1}(f)$ is not an $\mathcal{O}_{\mathbb{C}^{n}, 0}$-module, because $\mathcal{O}_{\mathbb{C}^{p}, 0}$ is not. It is, however, an $\mathcal{O}_{\mathbb{C}^{p}, 0}$-module; for via composition with $f, \mathcal{O}_{\mathbb{C}^{n}, 0}$ becomes an $\mathcal{O}_{\mathbb{C}^{p}, 0^{-}}$module: we can 'multiply' $g \in \mathcal{O}_{\mathbb{C}^{n}, 0}$ by $h \in \mathcal{O}_{\mathbb{C}^{p}, 0}$ using composition with $f$ to transport $h \in \mathcal{O}_{\mathbb{C}^{p}, 0}$ to $h \circ f \in \mathcal{O}_{\mathbb{C}^{n}, 0}$ :

$$
h \cdot g:=(h \circ f) g
$$

By this 'extension of scalars', every $\mathcal{O}_{\mathbb{C}^{n}, 0}$-module becomes an $\mathcal{O}_{\mathbb{C}^{p}, 0}$-module. This is where commutative algebra enters the picture. But we will not open the door to it in any serious way just yet. We simply note that

| $\theta_{\mathbb{C}^{p}, 0}$ | is an $\mathcal{O}_{\mathbb{C}^{p}, 0^{-} \text {-module }}$ |
| :--- | :--- |
| $\omega f: \theta_{\mathbb{C}^{p}, 0} \rightarrow \theta(f)$ | is $\mathcal{O}_{\mathbb{C}^{p}, 0^{-}}$-linear, so |
| $T^{1}(f)$ | is an $\mathcal{O}_{\mathbb{C}^{p}, 0}$-module |

### 3.2 First calculations

Example 3.16. (1) The map-germ

$$
f(x, y)=\left(x, y^{2}, x y\right)
$$

parametrising the cross-cap (pinch point, Whitney umbrella) is stable. We use coordinates ( $x, y$ ) on the source and $(X, Y, Z)$ on the target. We now calculate that $T^{1}(f)=0$. For this purpose we divide $\mathcal{O}_{\mathbb{C}^{2}, 0}$ into even and odd parts with respect to the $y$ variable, and denote them by $\mathcal{O}^{e}$ and $\mathcal{O}^{o}$. Every element of $\mathcal{O}^{e}$ can be written in the form $a\left(x, y^{2}\right)$, and every element of $\mathcal{O}^{o}$ in the form $y a\left(x, y^{2}\right)$. Then (we hope the notation is self-explanatory)

$$
\theta(f)=\left(\begin{array}{c}
\mathcal{O}^{e} \oplus \mathcal{O}^{o} \\
\mathcal{O}^{e} \oplus \mathcal{O}^{o} \\
\mathcal{O}^{e} \oplus \mathcal{O}^{o}
\end{array}\right)
$$

and since

$$
\omega f\left(\begin{array}{l}
a(X, Y)  \tag{3.23}\\
b(X, Y) \\
c(Y, Y)
\end{array}\right)=\left(\begin{array}{l}
a\left(x, y^{2}\right) \\
b\left(x, y^{2}\right) \\
c\left(x, y^{2}\right)
\end{array}\right)
$$

we see that the even part of $\theta(f)$ is indeed contained in $T^{1}(f)$, and we need worry only about the odd part. Since

$$
\operatorname{tf}\left(a\left(x, y^{2}\right) \partial / \partial x\right)=\left(\begin{array}{cc}
1 & 0  \tag{3.24}\\
0 & 2 y \\
y & x
\end{array}\right)\binom{a\left(x, y^{2}\right)}{0}=\left(\begin{array}{c}
a\left(x, y^{2}\right) \\
0 \\
y a\left(x, y^{2}\right)
\end{array}\right)
$$

we get all of the odd part of the third row. Since

$$
t f\left(a\left(x, y^{2}\right) \partial / \partial y\right)=\left(\begin{array}{cc}
1 & 0  \tag{3.25}\\
0 & 2 y \\
y & x
\end{array}\right)\binom{0}{a\left(x, y^{2}\right)}=\left(\begin{array}{c}
0 \\
2 y a\left(x, y^{2}\right) \\
x a\left(x, y^{2}\right)
\end{array}\right)
$$

we get all of the odd part of the second row. Since

$$
t f\left(y a\left(x, y^{2}\right) \partial / \partial x\right)=\left(\begin{array}{cc}
1 & 0  \tag{3.26}\\
0 & 2 y \\
y & x
\end{array}\right)\binom{y a\left(x, y^{2}\right)}{0}=\left(\begin{array}{c}
y a\left(x, y^{2}\right) \\
0 \\
y^{2} a\left(x, y^{2}\right)
\end{array}\right)
$$

we get all of the odd part of the first row. So $T \mathscr{A}_{e} f=\theta(f), T^{1}(f)=0$ and $f$ is stable.
(2) The map-germ $f(x, y)=\left(x, y^{2}, y^{3}+x^{2} y\right)$ is not stable. The calculation of (3.23), (3.25) and (3.26) still apply, with insignificant modifications. The only change from (1) is that (3.24) now shows that

$$
\begin{equation*}
T \mathscr{A}_{e} f \supset\left(x \mathcal{O}^{o}\right) \partial / \partial Z \tag{3.27}
\end{equation*}
$$

and we need an extra calculation

$$
t f\left(y a\left(x, y^{2}\right) \partial / \partial y\right)=\left(\begin{array}{ll}
1 & 0  \tag{3.28}\\
0 & 2 y \\
2 x y & x^{2}+3 y^{2}
\end{array}\right)\binom{0}{y a\left(x, y^{2}\right)}=\left(\begin{array}{l}
0 \\
2 y^{2} a\left(x, y^{2}\right) \\
x^{2} y a\left(x, y^{2}\right)+3 y^{3} a\left(x, y^{2}\right)
\end{array}\right)
$$



In view of (3.27) and what we know about the even terms, this completes the proof that

$$
T^{1}(f)=\left(\begin{array}{l}
\mathcal{O}^{e}+\mathcal{O}^{o}  \tag{3.29}\\
\mathcal{O}^{e}+\mathcal{O}^{o} \\
\mathcal{O}^{e}+x \mathcal{O}^{o}+y^{2} \mathcal{O}^{o}
\end{array}\right)
$$

It follows that $T^{1}(f)$ is generated, as a vector space over $\mathbb{C}$, by $y \partial / \partial Z$.
Definition 3.17. The $\mathscr{A}_{e}$-codimension of $f:\left(\mathbb{C}^{n}, 0\right) \rightarrow\left(\mathbb{C}^{p}, 0\right)$ is the dimension, as a $\mathbb{C}$-vector space, of $T^{1}(f)$.

Exercise 3.18. Calculate the $\mathscr{A}_{e}$-codimension, and a $\mathbb{C}$-basis for $T^{1}(f)$, when

1. $f(x, y)=\left(x, y^{2}, y^{3}+x^{k+1} y\right)$
2. $f(x, y)=\left(x, y^{2}, x^{2} y+y^{5}\right)$
3. $f(x, y)=\left(x, y^{2}, x^{2} y+y^{2 k+1}\right)$.

Remark 3.19. If $f:\left(\mathbb{C}^{2}, 0\right) \rightarrow\left(\mathbb{C}^{3}, 0\right)$ is not an immersion then the ideal $f^{*} m_{\mathbb{C}^{3}, 0}$ generated in $\mathcal{O}_{\mathbb{C}^{2}, 0}$ by the three component functions of $f$ is strictly contained in $m_{\mathbb{C}^{2}, 0}=(x, y)$. It follows that $\operatorname{dim}_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^{2}, 0} / f^{*} m_{\mathbb{C}^{3}, 0} \geq 2$. It can be shown ( $\left.\operatorname{cf}[36]\right)$ that every germ for which this dimension is exactly 2 (as in all the examples above) is $\mathscr{A}$-equivalent to one of the form $f(x, y)=\left(x, y^{2}, y p\left(x, y^{2}\right)\right)$. Alternative characterisation: these are the map-germs $\left(\mathbb{C}^{2}, 0\right) \rightarrow\left(\mathbb{C}^{3}, 0\right)$ of Boardman type $\sum^{1,0}$.

Question to ponder for later: what is the significance here of the involution $(x, y) \mapsto(x,-y)$ ?
Since we are nearly always referring to germs at 0 , write

$$
\begin{array}{rll}
\mathcal{O}_{n} & \text { in place of } & \mathcal{O}_{\mathbb{C}^{n}, 0} \\
\theta_{n} & \text { in place of } & \theta_{\mathbb{C}^{n}, 0} \\
m_{n} & \text { in place of } & m_{\mathbb{C}^{n}, 0}
\end{array}
$$

These examples are somewhat atypical. Calculating $T \mathscr{A}_{e} f$ is generally rather complicated. Checking that a given map-germ is it stable, however, is made much easier by a theorem of John Mather, which makes use of an auxiliary module known as the contact tangent space, and denoted $T \mathscr{K}_{e} f$, defined by

$$
T \mathscr{K}_{e} f=t f\left(\theta_{\mathbb{C}^{n}, 0}\right)+f^{*} m_{\mathbb{C}^{p}, 0} \theta(f) .
$$

Here $f^{*} m_{\mathbb{C}^{p}, 0}$ is the ideal in $\mathcal{O}_{\mathbb{C}^{n}, 0}$ generated by the component functions of $f$. When $p=1$, $T \mathscr{K}_{e} f$ is just the ideal $\left(f, \partial f / \partial x_{1}, \ldots, \partial f / \partial x_{n}\right)$ of $\mathcal{O}_{\mathbb{C}^{n}, 0}$. In any case it is always an $\mathcal{O}_{\mathbb{C}^{n}, 0}$-module,
which makes calculating with it very much easier than calculating $T \mathscr{A}_{e} f$. Like $T \mathscr{A}_{e} f, T \mathscr{K}_{e} f$ is the 'extended' tangent space to the orbit of $f$ under a group action, which we will not say anything about. The true tangent space here is $T \mathscr{K} f=t f\left(m_{n} \theta_{n}\right)+f^{*}\left(m_{p}\right) \theta(f)$.

Mather's theorem is
Theorem 3.20. If $T \mathscr{K}_{e} f+S p_{\mathbb{C}}\left\{\partial / \partial y_{1}, \ldots, \partial / \partial y_{p}\right\}=\theta(f)$ then $T^{1}(f)=0$ (so $f$ is stable).
Example 3.21. (1) We apply this theorem to the map-germ $f$ of Example 3.16(1). We have

$$
\begin{aligned}
T \mathscr{K}_{e} f & =t f\left(\theta_{\mathbb{C}^{2}, 0}\right)+f^{*} m_{\mathbb{C}^{3}, 0} \theta(f) \\
& =\mathcal{O}_{\mathbb{C}^{2}, 0} \cdot\{\partial f / \partial x, \partial f / \partial y\}+\left(x, y^{2}\right) \theta(f) \\
& =\mathcal{O}_{\mathbb{C}^{2}, 0} \cdot\left\{\left(\begin{array}{c}
1 \\
0 \\
y
\end{array}\right),\left(\begin{array}{c}
0 \\
2 y \\
x
\end{array}\right)\right\}+\left(\begin{array}{c}
\left(x, y^{2}\right) \\
\left(x, y^{2}\right) \\
\left(x, y^{2}\right)
\end{array}\right)
\end{aligned}
$$

You can easily show that the condition of the theorem holds; in particular, since ( $x, y^{2}$ ) contains the square of the maximal ideal of $\mathcal{O}_{\mathbb{C}^{2}, 0}$, it's necessary only to check for terms of degree 0 and 1.
(2) The same theorem can be used to show that the map-germs

1. $f:\left(\mathbb{C}^{3}, 0\right) \rightarrow\left(\mathbb{C}^{3}, 0\right)$ defined by

$$
f\left(x_{1}, x_{2}, x_{3}\right)=\left(x_{1}, x_{2}, x_{3}^{4}+x_{1} x_{3}^{2}+x_{2} x_{3}\right)
$$

2. $f:\left(\mathbb{C}^{4}, 0\right) \rightarrow\left(\mathbb{C}^{5}, 0\right)$ defined by

$$
f\left(x_{1}, x_{2}, x_{3}, x_{4}\right)=\left(x_{1}, x_{2}, x_{3}, x_{4}^{3}+x_{1} x_{4}, x_{2} x_{4}^{2}+x_{3} x_{4}\right)
$$

3. $f:\left(\mathbb{C}^{5}, 0\right) \rightarrow\left(\mathbb{C}^{6}, 0\right)$ defined by

$$
f(x, y, a, b, c, d)=\left(x^{2}+a y, x y+b x+c y, y^{2}+d x, a, b, c, d\right)
$$

are stable. These are left as Exercises.
Remark 3.22. The reader will note that each of the germs listed in Example 3.21(2) is itself an unfolding of a germ of rank 0 (i.e. whose derivative at 0 vanishes). Of course, by means of the inverse function theorem any germ can be put in this form, in suitable coordinates. But in fact there is a general procedure for finding all stable map-germs as unfoldings of lower-dimensional germs of rank zero, based on Mather's theorem quoted here. The procedure is the following:

1. Given $f:\left(\mathbb{C}^{n}, 0\right) \rightarrow\left(\mathbb{C}^{p}, 0\right)$ of rank 0 , calculate $T \mathscr{K}_{e} f$, and find a basis for the quotient $\theta(f) / T \mathscr{K}_{e} f$.
2. If $g_{1}, \ldots, g_{d} \in \theta(f)$ project to a basis for the quotient $\theta(f) / T \mathscr{K}_{e} f$ then the unfolding $F$ : $\left(\mathbb{C}^{n} \times \mathbb{C}^{d},(0,0)\right) \rightarrow\left(\mathbb{C}^{p} \times \mathbb{C}^{d},(0,0)\right)$ defined by

$$
\begin{equation*}
F\left(x, u_{1}, \ldots, u_{d}\right)=\left(f(x)+\sum_{j} u_{j} g_{j}(x), u_{1}, \ldots, u_{d}\right) \tag{3.30}
\end{equation*}
$$

is stable.
3. If among the basis elements $g_{1}, \ldots, g_{d}$ we have any constant term, of the form

$$
g_{j}=(0, \ldots, 0,1,0, \ldots, 0)^{t}
$$

then omitting $g_{j}$ and its corresponding unfolding parameter $u_{j}$ from the formula (3.30), we still obtain a stable map-germ. Adding $u_{j} g_{j}$ in (3.30) merely contributes a trivial unfolding.

Exercise 3.23. Apply this procedure starting with $f(x, y)=\left(x^{2}, y^{2}\right)$.
An ingenious result, due to Terry Gaffney, and extending Mather's, allows one to transform a guess for $T \mathscr{A}_{e} f$, (based perhaps on a calculation modulo some power of the maximal ideal (i.e. ignoring all terms of degree higher than some fixed $k$ )) into a rigorous calculation.

Theorem 3.24. Suppose that $f:\left(\mathbb{C}^{n}, 0\right) \rightarrow\left(\mathbb{C}^{p}, 0\right)$ is a map-germ such that

$$
T \mathscr{K}_{e} f \supset \mathfrak{m}_{\mathbb{C}^{n}, 0}^{\ell} \theta(f)
$$

and $C \subset \theta(f)$ is an $\mathcal{O}_{\mathbb{C}^{p}, 0}$-submodule such that

$$
C \supset \mathfrak{m}_{\mathbb{C}^{n}, 0}^{k} \theta(f)
$$

(where $k>0$ ). Then

$$
C=T \mathscr{A}_{e} f \quad \Leftrightarrow \quad C=T \mathscr{A}_{e} f+f^{*} \mathfrak{m}_{\mathbb{C}^{p}, 0} C+\mathfrak{m}_{\mathbb{C}^{n}, 0}^{k+\ell} \theta(f) .
$$

The proof I know of is in [36, 3:2]
Exercise 3.25. Find the smallest integer $\ell$ such that $T \mathscr{K}_{e} f \supset \mathfrak{m}_{2}^{\ell} \theta(f)$ when $f$ is the map germ of Example 3.16(2).

### 3.3 Consequences of Finite Codimension

Let $f: \mathbb{C}^{n} \rightarrow \mathbb{C}^{p}\left(\right.$ or $\mathbb{R}^{n} \rightarrow \mathbb{R}^{p}$ ) be an analytic (or $C^{\infty}$ ) map. Its $k$-jet at a point $x$ is the $p$ tuple consisting of the Taylor polynomials of degree $k$ of its component functions. The $k$-jet of $f$ at $x$ is denoted by $j^{k} f(x)$. We say that a map-germ $f:\left(\mathbb{C}^{n}, x\right) \rightarrow\left(\mathbb{C}^{p}, y\right)$ is $k$-determined for $\mathscr{A}$-equivalence if any other map-germ having the same $k$-jet at $x$ is $\mathscr{A}$-equivalent to $f$, and finitely determined for $\mathscr{A}$-equivalence if this holds for some finite value of $k$.
Theorem 3.26. (J.Mather [29]) $f$ is finitely determined if and only if $\operatorname{dim}_{\mathbb{C}} T^{1}(f)<\infty$.
The smallest value of $k$ for which this holds is the determinacy degree of $f$. Finding good estimates for the determinacy degree of $f$ in terms of easily calculable data was once a major endeavour. Mather's original estimates (in [29]) were impractically large. They were greatly improved by Terry Gaffney and Andrew du Plessis ([14], [11]). In particular the following estimate due to Gaffney is useful:
Theorem 3.27. ([14]) If $T \mathscr{A}_{e} f \supset \mathfrak{m}_{\mathbb{C}^{n}, 0}^{k} \theta(f)$ and $T \mathscr{K}_{e} f \supset \mathfrak{m}_{\mathbb{C}^{n}, 0}^{\ell} \theta(f)$ then $f$ is $k+\ell$-determined.
Since we are reaching conclusions about the $\mathscr{A}$-orbit of $f$, it is slightly curious that our hypotheses are framed in terms of $T \mathscr{A}_{e} f$ and not $T \mathscr{A} f$. Indeed it is (almost) obvious that if $f$ is $k$-determined then

$$
\begin{equation*}
T \mathscr{A} f \supset \mathfrak{m}_{n}^{k+1} \theta(f) \tag{3.31}
\end{equation*}
$$

To make this clear, we introduce the jet spaces $J^{k}(n, p)$.

Definition 3.28. 1. $\mathfrak{m}(n, p)$ is the vector space of all germs $\left(\mathbb{C}^{n}, 0\right) \rightarrow\left(\mathbb{C}^{p}, 0\right)$. It can be identified with $\mathfrak{m}_{n} \theta(f)$ for any $f \in \mathcal{O}(n, p)$.
2. $J^{k}(n, p)$ is the set of $k$-jets of germs $\left(\mathbb{C}^{n}, 0\right) \rightarrow\left(\mathbb{C}^{p}, 0\right)$.
3. $j^{k}: \mathcal{O}(n, p) \rightarrow J^{k}(n, p)$ is the operation "take the $k$-jet". The map $j^{k}: \mathcal{O}(n, p) \rightarrow J^{k}(n, p)$ is surjective. Its kernel is $\mathfrak{m}_{n}^{k} \mathfrak{m}(n, p)$, so we can view $J^{k}(n, p)$ as $\mathfrak{m}(n, p) / \mathfrak{m}_{n}^{k} \mathfrak{m}(n, p)$.
4. For $k \leq \ell, \pi_{k}^{\ell}: J^{\ell}(n, p) \rightarrow J^{k}(n, p)$ is the projection ("truncate at degree $k$ ")
5. $\mathscr{A}^{k}=j^{k}(\mathscr{A}) \subset J^{k}(n, n) \times J^{k}(p, p)$ is the quotient of $\mathscr{A}$ acting naturally on $J^{k}(n, p)$.

The diagram (in which the rows are group actions)

is commutative. The lower row is a finite-dimensional model of the upper row. In the lower row we really do have an algebraic group acting algebraically on an algebraic variety - indeed, on a finite dimensional complex vector space. This model provides motivation for many assertions, such as the statement that if $f$ is $k$-determined then $T \mathscr{A} f \supset \mathfrak{m}_{n}^{k+1} \theta(f)$. What is clear is that if $f$ is $k$-determined then

$$
\mathscr{A}^{(\ell)} j^{\ell} f(0)=\left(\pi_{k}^{\ell}\right)^{-1}\left(\mathscr{A}^{(k)} j^{k} f(0)\right) .
$$

Now $\pi_{k}^{\ell}$ is linear, and its kernel is $j^{\ell}\left(\mathfrak{m}^{k+1} \theta(f)\right)$. So if $f$ is $k$-determined,

$$
T \mathscr{A}^{(\ell)} j^{\ell} f(0) \supset j^{\ell}\left(m^{k+1} \theta(f)\right)
$$

Since

$$
J^{\ell}(n, p)=\mathfrak{m}_{n} \theta(f) / \mathfrak{m}_{n}^{\ell+1} \theta(f),
$$

this can be rewritten

$$
\begin{equation*}
T \mathscr{A} f+\mathfrak{m}_{n}^{\ell+1} \theta(f) \supset \mathfrak{m}_{n}^{k+1} \theta(f), \tag{3.33}
\end{equation*}
$$

almost the statement (3.31) described as obvious above. If we knew that $\mathfrak{m}_{n}^{k+1} \theta(f)$ were a finitely generated module over $\mathcal{O}_{\mathbb{C}^{p}, 0}$ then an application of Nakayama's Lemma would prove (3.31). But we don't know it, and in fact if $n>p$ it can't be true. Neverthless, it is possible to deduce (3.31) from (3.33) using some algebraic/analytic geometry:

1. $T \mathscr{K}_{e} f \supset T \mathscr{A} f$, so (3.33) implies

$$
\begin{equation*}
T \mathscr{K}_{e} f+\mathfrak{m}_{n}^{\ell+1} \theta(f) \supset \mathfrak{m}_{n}^{k+1} \theta(f) . \tag{3.34}
\end{equation*}
$$

2. Because (3.34) involves only $\mathcal{O}_{\mathbb{C}^{n}, 0}$-modules, by Nakayama's Lemma we deduce that $T \mathscr{K}_{e} f \supset$ $\mathfrak{m}_{n}^{k+1} \theta(f)$. This implies that $\operatorname{dim}_{\mathbb{C}}\left(\theta(f) / T \mathscr{K}_{e} f\right)<\infty(f$ is " $\mathscr{K}$-finite", or has "finite singularity type".)
3. Let $J_{f}$ be the ideal in $\mathcal{O}_{\mathbb{C}^{n}, 0}$ generated by the $p \times p$ minors of the matrix of $d f$. Its locus of zeros is the critical set $\sum_{f}$, the set of points where $f$ is not a submersion. By taking the determinants of $p$-tuples of elements of $\theta(f)$, from the fact that $f$ is $\mathscr{K}$ finite we deduce that $\operatorname{dim}_{\mathbb{C}}\left(\mathcal{O}_{\mathbb{C}^{n}, 0} / J_{f}+f^{*} \mathfrak{m}_{p} \mathcal{O}_{\mathbb{C}^{n}, 0}\right)<\infty$. This condition has a clear geometrical significance (over the complex numbers!):

$$
V\left(J_{f}+f^{*} \mathfrak{m}_{p} \mathcal{O}_{\mathbb{C}^{n}, 0}\right)=\sum_{f} \cap f^{-1}(0),
$$

so $f$ is finite-to-one on its critical locus.
4. From this it follows that every coherent sheaf of $\mathcal{O}_{\mathbb{C}^{n}, 0}$ modules supported on $\sum_{f}$ is finite over $\mathcal{O}_{\mathbb{C}^{p}, 0}$. In particular

$$
\left(\mathfrak{m}^{\ell+1} \theta(f)+t f\left(\theta_{n}\right)\right) / t f\left(\theta_{n}\right)
$$

is a finite $\mathcal{O}_{\mathbb{C}^{p}, 0}$-module! So now we can apply Nakayama's Lemma to deduce (3.31) from (3.33): simply take the quotient on both sides by $t f\left(\theta_{n}\right)$.

It took some quite non-elementary steps to get to the "obvious" statement (3.31) from the truly obvious statement (3.33)!

Exercise 3.29. Use the techniques just introduced to prove Theorem 3.20. Note that the hypothesis of 3.20 is equivalent to

$$
\theta(f)=T \mathscr{A}_{e} f+T \mathscr{K}_{e} f=T \mathscr{A}_{e} f+f^{*} \mathfrak{m}_{p} \theta(f) .
$$

In view of the fact that (3.31) is true, one might hope that its converse, which also seems reasonable, should also be true. But things are not so simple. They become simpler if we replace the group $\mathscr{A}$ by its subgroup $\mathscr{A}_{1}$ consisting of pairs of germs of diffeomorphisms whose derivative at 0 is the identity. This observation by Bill Bruce led to what was probably the final paper on finite determinacy, [1], in which unipotent groups $\mathscr{G}$ are identified as those for which the determinacy degree is equal to one less than the smallest power $k$ such that $m_{n}^{k} \theta(f) \subseteq T \mathscr{G}_{e} f$. The group $\mathcal{A}$ itself is not unipotent.

To prove a statement of the kind

$$
T \mathcal{A} f \supset \mathfrak{m}_{n}^{k} \theta(f) \quad \Longrightarrow \quad f \text { is } d(k) \text {-determined }
$$

one has to show that if $g$ and $f$ differ by terms in $\mathfrak{m}_{n}^{d(r)+1}$ then two things happen:

1. first, the germ of deformation $f+t(g-f)$ is trivial - so that for all $t$ is some neighbourhood of $0, f+t(g-f)$ is equivalent to $f$.
2. Second, that for any value $t_{0}$ of $t$, we also have $T \mathcal{A}\left(f+t_{0}(g-f)\right) \supset \mathfrak{m}_{n}^{k} \theta(f)-$ so that by the first assertion, the deformation $f+t g$ is trivial also in the neighbourhood of $t_{0}$.

In practice, one should not expect to obtain the precise determinacy degree of a map-germ from a general theorem like 3.27. Instead, one can often significantly improve an estimate by using another result due to Mather (in [29, lemma]) and known as "Mather's Lemma".

Proposition 3.30. Suppose the Lie group $G$ acts smoothly on the manifold $M$, and that $W \subset M$ is a smooth connected submanifold. Then a necessary and sufficient condition for $W$ to be contained in a single orbit is that

1. for all $x \in W, T_{x} W \subset T_{x} G x$, and
2. the dimension of $T_{x} G x$ is the same for all $x \in W$.

One uses the lemma as follows: suppose that it is possible to show, e.g. by applying a general theorem, that $f$ is $\ell$-determined, and wants to show that it is $k$-determined for some $k<\ell$. Let $M=J^{\ell}(n, p), G=\mathcal{A}^{(\ell)}$ and

$$
W=\left\{j^{\ell} g: j^{k} g=j^{k} f\right\}
$$

Exercise 3.31. If $W$ lies in a single $\mathcal{A}^{(\ell)}$-orbit then $f$ is $k$-determined.
Because we are working modulo $\mathfrak{m}^{\ell+1}$, terms of degree $\ell+1$ and higher can be ignored in calculating $T \mathcal{A}^{(\ell)} g$, and this may make it relatively straightforward to show that the conditions of Mather's Lemma hold.

### 3.4 Multi-germs

We have spoken only of 'mono'-germs $\left(\mathbb{C}^{n}, 0\right) \rightarrow\left(\mathbb{C}^{p}, 0\right)$. But many of the interesting phenomena associated with deformations of mono-germs require description in terms of multi-germs, so they cannot sensibly be avoided. For example, a parametrised plane curve singularity splits into a certain number of nodes on deformation; each of these is stable; their number is an important invariant of the singularity.


Figure 1: $t \mapsto\left(t^{2}, t^{7}\right)$

$$
t \mapsto\left(t^{2}, t\left(t^{2}-4 u\right)\left(t^{2}-9 u\right)\left(t^{2}-16 u\right)\right)
$$

Example 3.32. The bi-germ consisting of two germs of immersion from $\mathbb{C}$ to $\mathbb{C}^{2}$ which meet tangentially is not stable. In suitable coordinates such a germ can be written

$$
\left\{\begin{array}{l}
f^{(1)}: s \mapsto(s, 0)  \tag{3.35}\\
f^{(2)}: t \mapsto(t, h(t))
\end{array}\right.
$$

We use independent coordinate systems $s, t$ centred on each of the base-points. It will be useful to label the base-points $0^{(1)}$ and $0^{(2)}$. The two branches meet tangentially if $h \in\left(t^{2}\right)$. Let us calculate $T^{1}(f)$. We have

$$
\begin{gathered}
\theta(f)=\theta\left(f^{(1)}\right) \oplus \theta\left(f^{(2)}\right) \\
t f: \theta_{\mathbb{C},\left\{0^{(1)}, 0^{(2)}\right\}} \rightarrow \theta(f) \quad \text { is equal to } \quad t f^{(1)} \oplus t f^{(2)} \\
\theta_{\mathbb{C}^{2}, 0} \rightarrow \theta(f) \quad \text { is given by } \quad \eta \mapsto\left(\eta \circ f^{(1)}, \eta \circ f^{(2)}\right)
\end{gathered}
$$



We represent elements of $\theta(f)$ as $2 \times 2$-matrices, in which the first column is in $\theta\left(f^{(1)}\right)$ and the second in $\theta\left(f^{(2)}\right)$. Elements of $\theta_{\mathbb{C},\left\{0^{(1)}, 0^{(2)}\right\}}$ are written as pairs $(a(s) \partial / \partial s, b(t) \partial / \partial t)$. Then

$$
t f(a(s) \partial / \partial s, 0)=\left[\begin{array}{cc}
a(s) & 0  \tag{3.36}\\
0 & 0
\end{array}\right]
$$

so in $T \mathscr{A}_{e} f$ we have everything in the top left corner; also

$$
\begin{gather*}
t f(0, b(t) \partial / \partial t)=\left[\begin{array}{cc}
0 & b(t) \\
0 & h^{\prime}(t) b(t)
\end{array}\right]  \tag{3.37}\\
\omega f\left(\left[\begin{array}{l}
\eta_{1} \\
\eta_{2}
\end{array}\right]\right)=\left[\begin{array}{cc}
\eta_{1}(s, 0) & \eta_{1}(t, h(t)) \\
\eta_{2}(s, 0) & \eta_{2}(t, h(t))
\end{array}\right] . \tag{3.38}
\end{gather*}
$$

Using (3.38) with $\eta_{2}=0$, in view of (3.36) we get everything in the top right corner. Now using (3.37), in the bottom right hand corner we get everything in the Jaocbian ideal $J_{h}$, and using (3.38) with $\eta_{1}=0$ and $\eta_{2}(X, Y)=p(X)$ we get everything of the form

$$
\left[\begin{array}{cc}
0 & 0 \\
p(s) & p(t)
\end{array}\right] .
$$

We have essentially shown

## Proposition 3.33.

$$
\theta(f) / T \mathscr{A}_{e} f \simeq \mathcal{O}_{\mathbb{C}, 0^{(2)}} / J_{h}
$$

Notice that $f$ can be perturbed to a bi-germ with $\nu$ nodes, where $\nu$ is the order of $h$. So the number of nodes is one more than the codimension. The relation between the $\mathcal{A}_{e}$-codimension of a map-germ and the geometry and topology of a stable perturbation is one of the most interesting aspects of the subject, and will be explored further below.

### 3.5 Finite codimension equals isolated instability

The next theorem is stated int two parts; the first is a special case of the second, but is easier to make sense of.

Theorem 3.34. (Terry Gaffney) (1) $f:\left(\mathbb{C}^{n}, S\right) \rightarrow\left(\mathbb{C}^{p}, 0\right)(n<p)$ has finite $\mathscr{A}_{e}$-codimension if and only if for every representative $f: U \rightarrow V$ of $f$ there is a neighbourhood $V_{0}$ of $0 \in V$ such that for every $y \in V_{0} \backslash\{0\}$ the multi-germ $f:\left(\mathbb{C}^{n}, f^{-1}(y)\right) \rightarrow\left(\mathbb{C}^{p}, y\right)$ is stable.
(2) $f:\left(\mathbb{C}^{n}, S\right) \rightarrow\left(\mathbb{C}^{p}, 0\right)(n \geq p)$ has finite $\mathscr{A}_{e}$-codimension if and only if for every representative $f: U \rightarrow V$ of $f$ there is a neighbourhood $V_{0}$ of $0 \in V$ such that for every $y \in V_{0} \backslash\{0\}$ the multi-germ $f:\left(\mathbb{C}^{n}, f^{-1}(y) \cap \sum_{f}\right) \rightarrow\left(\mathbb{C}^{p}, y\right)$ is stable.
 singularity


Sufficiently small deformation


Too large deformation!

This theorem is an easy application of the theory of coherent analytic sheaves; there is a a proof in [49]. As a consequence of 3.34 , when a germ of finite codimension is deformed, the only qualitative changes occur in the vicinity of the unique unstable point. Near the boundary of the domain of any representative of the germ, nothing changes, in a sufficiently small deformation.

### 3.6 Versal Unfoldings

An unfolding of a map-germ $f_{0}$ is $\mathcal{A}_{e}$-versal if it contains, up to parametrised $\mathcal{A}$-equivalence, every possible unfolding of the germ. In this section we make precise sense of this idea, and study some examples.

Definition 3.35. (1) Let $F, G:\left(\mathbb{C}^{n} \times \mathbb{C}^{d}, 0\right) \rightarrow\left(\mathbb{C}^{p} \times \mathbb{C}^{d}, 0\right)$ be unfoldings of the same map germ $f_{0}:\left(\mathbb{C}^{n}, 0\right) \rightarrow\left(\mathbb{C}^{p}, 0\right)$. They are equivalent if there exist germs of diffeomorphisms

$$
\Phi:\left(\mathbb{C}^{n} \times \mathbb{C}^{d}, 0\right) \rightarrow\left(\mathbb{C}^{n} \times \mathbb{C}^{d}, 0\right)
$$

and

$$
\Psi:\left(\mathbb{C}^{p} \times \mathbb{C}^{d}, 0\right) \rightarrow\left(\mathbb{C}^{p} \times \mathbb{C}^{d}, 0\right)
$$

such that

1. $\Phi(x, u)=(\varphi(x, u), u)$ and $\varphi(x, 0)=x$
2. $\Psi(y, h)=(\psi(y, u), u)$ and $\psi(y, 0)=y$
3. $F=\Psi \circ G \circ \Phi$

Note that an unfolding is trivial (Definition 3.11) if it is equivalent to the constant unfolding.
(2) With $F(x, u)=(f(x, u), u)$ as in (1), let $h:\left(\mathbb{C}^{e}, 0\right) \rightarrow\left(\mathbb{C}^{d}, 0\right)$ be a map germ. The unfolding $\left(\mathbb{C}^{n} \times \mathbb{C}^{e}, 0\right) \rightarrow\left(\mathbb{C}^{p} \times \mathbb{C}^{e}, 0\right)$ defined by

$$
(x, v) \mapsto(f(x, h(v)), v)
$$

is called the pull-back of $F$ by $h$, and denoted by $h^{*} F$. The map-germ $h$ in this context is often called the 'base-change' map, and we say that $h^{*} F$ is the unfolding induced from $F$ by $h$.
(3) The unfolding $F$ of $f_{0}$ is $\mathcal{A}_{e}$-versal if for every other unfolding $G:\left(\mathbb{C}^{n} \times \mathbb{C}^{e}, 0\right) \rightarrow\left(\mathbb{C}^{p} \times \mathbb{C}^{e}, 0\right)$ of $f_{0}$, there is a base-change map $h:\left(\mathbb{C}^{e}, 0\right) \rightarrow\left(\mathbb{C}^{d}, 0\right)$ such that $G$ is equivalent (in the sense of (1)) to the unfolding $h^{*} F$ (as defined in (2)).


The term 'versal' if the intersection of the words 'universal' and 'transversal'. Versal unfoldings were once upon a time called universal, but later it was decided that they did not deserve this term, because the base-change map $h$ of part (3) of the definition is not in general unique. Uniqueness is an important ingredient in the "universal properties" which characterise many mathematical objects, and so universal unfoldings were stripped of their title. However the intersection with the word 'transversal' is serendipitous, as we will see.
Example 3.36. Some light relief Consider a manifold $M \subset \mathbb{C}^{N}$. Radial projection from a point $q$ into a hyperplane $H$ is defined by the following picture: It defines a map $P_{q}: M \rightarrow H$. If the hyperplane $H$ is replaced by another hyperplane $H^{\prime}$, then the corresponding projection $P_{q}^{\prime}: M \rightarrow H^{\prime}$ is left-equivalent to $P_{q}$; composing $P_{q}^{\prime}$ with the restriction of $P_{q}$ to $H^{\prime}$, we get $P_{q}$. On the other hand, if we vary the point $q$ then we may well deform the projection $P_{q}$ non-trivially. So we consider the unfolding

$$
P: M \times \mathbb{C}^{N} \rightarrow H \times \mathbb{C}^{N} .
$$

It's instructive to look at this over $\mathbb{R}$ with the help of a piece of bent wire and an overhead projector. Are the unstable map-germs one sees versally unfolded in the family of all projections? This is discussed in [48] and again in [38].

Like stability, versality can be checked by means of an infinitesimal criterion. Let $F(x, u)=$ $(f(x, u), u)$ be an unfolding of $f_{0}$. Write $\partial f /\left.\partial u_{j}\right|_{u=0}$ as $\dot{F}_{j}$.
Theorem 3.37. (Infinitesimal versality is equivalent to versality) The unfolding $F$ of $f_{0}$ is versal if and only if

$$
T \mathscr{A}_{e} f_{0}+S p_{\mathbb{C}}\left\{\dot{F}_{1}, \ldots, \dot{F}_{d}\right\}=\theta\left(f_{0}\right)
$$

- in other words, if the images of $\dot{F}_{1}, \ldots, \dot{F}_{d}$ in $T^{1}\left(f_{0}\right)$ generate it as (complex) vector space.

For a proof, see Chapter X of Martinet's book [27]. Martinet proves the theorem for $C^{\infty}$ map-germs; the proof in the analytic category is the same. Both use the Preparation Theorem, 2.30 .

Exercise 3.38. Prove 'only if' in Theorem 3.37. It follows in a straightforward way from the definitions: let $g$ be an arbitrary element of $\theta\left(f_{0}\right)$ and take, as $G$, the 1-parameter unfolding

$G(x, t)=(f(x)+t g(x), t)$. Show that if $G$ is equivalent to an unfolding induced from $F$ then $g \in T \mathscr{A}_{e} f_{0}+\operatorname{Sp}_{\mathbb{C}}\left\{\dot{F}_{1}, \ldots, \dot{F}_{d}\right\}$
Example 3.39. Consider the map-germ $f_{0}(x, y)=\left(x, y^{2}, y^{3}+x^{2} y\right)$ of Example 3.16. We saw that $y \partial / \partial Z$ projects to a basis for $T^{1}\left(f_{0}\right)$. So

$$
F(x, y, u)=\left(x, y^{2}, y^{3}+x^{2} y+u y, u\right)
$$

is a versal deformation. What is the geometry here? Think of $F$ as a family of mappings,

$$
f_{u}(x, y)=\left(x, y^{2}, y^{3}+x^{2} y+u y\right) .
$$

The ramification ideal $\mathscr{R}_{f_{u}} \subset \mathcal{O}_{\mathbb{C}^{2}}$ generated by the $2 \times 2$ minors of the matrix $\left[d f_{u}\right]$ defines the set of points where $f_{u}$ fails to be an immersion. Here $\mathscr{R}_{f_{u}}=\left(y, x^{2}+u\right)$. So for $u \neq 0, f_{u}$ has two non-immersive points. They are only visible over $\mathbb{R}$ when $u<0$. How does $f_{u}$ behave in the neighbourhood of each of these points? At each, $\mathscr{R}_{f_{u}}$ is equal to the maximal ideal; it follows that $d f_{u}$ is transverse to the submanifold $\sum^{1} \subset L\left(\mathbb{C}^{2}, \mathbb{C}^{3}\right)$ consisting of linear maps of rank 1 . In fact this transversality characterises the map-germ $f$ of $3.16(1)$ up to $\mathscr{A}$-equivalence, though here we are not yet able to show that. Using this characterisation, we see that in a neighbourhood of the image of each of the two points $( \pm \sqrt{-u}, 0)$, the image of $f_{u}$ looks like the drawing in Example 3.16. The key to assembling the image of $f_{u}$ from its constituent parts is the curve of self-intersection. The only points mapped 2-1 by $f_{u}$ are the points of the curve $\left\{x^{2}+y^{2}+u=0\right\}$; for $u<0$ this is a circle when viewed over $\mathbb{R}$. Here points $(x, \pm y)$ share the same image. The two non-immersive points of $f_{u}$ are the fixed points of the involution $(x, y) \mapsto(x,-y)$ which interchanges pairs of points sharing the same image.

The image contains a chamber; indeed it is homotopy-equivalent to a 2 -sphere. This is no coincidence. The next figure shows images of stable perturbations of each of the remaining codimension 1 singularities of maps from surfaces into 3 -space. Each is homotopy-equivalent to a 2 -sphere. Some choices have been made regarding the real form: sometimes a change of sign which makes no difference over $\mathbb{C}$ does make a difference over $\mathbb{R}$. Nevertheless in all of these cases it is possible to choose a suitable real form whose perturbation is a homotopy 2 -sphere.

Exercise 3.40. Find versal unfoldings of the following germs:

1. $f:(\mathbb{C}, 0) \rightarrow\left(\mathbb{C}^{2}, 0\right), f(t)=\left(t^{3}, t^{4}\right)$.
2. $f:(\mathbb{C}, 0) \rightarrow\left(\mathbb{C}^{2}, 0\right), f(t)=\left(t^{2}, t^{5}\right)$.
3. $f:(\mathbb{C}, 0) \rightarrow\left(\mathbb{C}^{2}, 0\right), f(t)=\left(t^{2}, t^{2 k+1}\right)$.

Figure 2: Images of stable perturbations of codimension 1 germs of maps from the plane to 3 -space

4. $f:\left(\mathbb{C}^{2}, 0\right) \rightarrow\left(\mathbb{C}^{3}, 0\right), f(x, y)=\left(x, y^{2}, y^{3}+x^{k+1} y\right)$.
5. $f:\left(\mathbb{C}^{2}, 0\right) \rightarrow\left(\mathbb{C}^{2}, 0\right), f(x, y)=\left(x, y^{3}+x^{2} y\right)$.

### 3.7 Stable perturbations

We have looked at examples of mappings from $\mathbb{C}^{n}$ to $\mathbb{C}^{n+1}$ for $n=1,2$. By inspection, we can see that the perturbations of the unstable maps we considered were at least locally stable: every (monoand multi-) germ they contain is stable. In the dimension range we have looked at, every germ of finite codimension can be perturbed so that it becomes stable. These are "nice dimensions", to use a term due to John Mather. These dimension-pairs may be characterised by the following property: in the base of a versal deformation, the set of parameter-values $u$ such that $f_{u}$ has an unstable multi-germ is a proper analytic subvariety. It is known as the bifurcation set.

Mather carried out long calculations to determine the nice dimensions, published in [33]. $\mathrm{Cu}-$ riously, the nice dimensions are also characterised by the fact that every stable germ in these dimensions is weighted homogeneous, in appropriate coordinates.

When the bifurcation set $B$ is a proper analytic subvariety of a smooth space, it does not separate it topologically (remember we're working in $\mathbb{C}^{d}$ ). That is, any two points $u_{1}$ and $u_{2}$ in its complement can be joined by a path $\gamma(t)$ which does not meet $B$. Because $f_{u_{1}}$ and $f_{u_{2}}$ are locally stable, each germ of the unfolding

$$
(x, t) \mapsto\left(f_{\gamma(t)}(x), t\right)
$$

is trivial; so $f_{u_{1}}$ and $f_{u_{2}}$ are locally isomorphic and globally $C^{\infty}$-equivalent. Thus, to each complex germ of finite codimension we can associate a stable perturbation (any one of the mappings $f_{u}$ for $u \notin B)$ which is independent of the choice of $u$, at least up to diffeomorphism. Some care must be taken to define the domain of $f_{u}$; it is more than a germ, but not a global mapping $\mathbb{C}^{n} \rightarrow \mathbb{C}^{p}$. The situation is analogous to the construction of the Milnor fibre, in which several choices of neighbourhoods must be made, but in which the final result is nevertheless independent of the choices. Details may be found in [25].

## 4 Stable Images and Discriminants

### 4.1 Review of the Milnor fibre

In the theory of isolated hypersurface singularities a key role is played by the Milnor fibre. Here is a very brief description.

1. Let $f$ be a complex analytic function defined on some neighbourhood of 0 in $\mathbb{C}^{n+1}$, and suppose it has isolated singularity at 0 . Then by the curve selection lemma, there exists $\varepsilon>0$ such that for $\varepsilon^{\prime}$ with $0<\varepsilon^{\prime} \leq \varepsilon$, the sphere of radius $\varepsilon^{\prime}$ centred at 0 is transverse to $f^{-1}(0)$. Let $B_{\varepsilon}$ be the closed ball centred at 0 and with radius $\varepsilon$. Then from the transversality it follows that $f^{-1}(0) \cap B_{\varepsilon}(0)$ is homeomorphic (indeed, diffeomorphic except at 0 ) to the cone on its boundary $f^{-1}(0) \cap S_{\varepsilon}$. The ball $B_{\varepsilon}(0)$ is a Milnor ball for the singularity.
2. By an argument involving properness, one can show that for suitably small $\eta>0$, all fibres $f^{-1}(t)$ with $|t|<\eta$ are transverse to $S_{\varepsilon}$. Let $D_{\eta}$ be the closed ball in $\mathbb{C}$ with radius $\eta$ and centre 0 , and let $D_{\eta}^{*}=D_{\eta} \backslash\{0\}$.
3. By the Ehresmann fibration theorem,

$$
f \mid: B_{\varepsilon} \cap f^{-1}\left(D_{\eta}^{*}\right) \rightarrow D_{\eta}^{*}
$$

is a $C^{\infty}$-locally trivial fibration. It is known as the Milnor fibration. Up to fibre-homeomorphism, it is independent of the choice of $\varepsilon$.
4. Its fibre is called the Milnor fibre of $f$. It has the homotopy type of a wedge of $n$-spheres, whose number $\mu$, the Milnor number of $f$, is equal to the dimension of the Jacobian algebra of $f$,

$$
\mathcal{O}_{\mathbb{C}^{n+1}, 0} / J_{f}
$$

The argument for the last statement is based on two facts:

1. if $\operatorname{dim}_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^{n+1}, 0} / J_{f}=1$ (in which case $f$ is said to have a 'non-degenerate" critical point), then by the holomorphic Morse lemma, $f$ is right-equivalent to $x \mapsto x_{1}^{2}+\cdots+x_{n+1}^{2}$. An explicit calculation now shows that the Milnor fibre is diffeomorphic to the unit ball sub-bundle of the tangent bundle of $S^{n}$. This has $S^{n}$ as a deformation-retract.
2. $f$ can be perturbed so that the critical point at 0 splits into non- degenerate critical points. There are exactly $\mu$ of them, and each contributes one sphere to the wedge.

The dimension of the Jacobian algebra plays a second, completely different, role in the theory. The quotient by which we measure instability,

$$
\frac{\left\{\left.\frac{d}{d t} f_{t}\right|_{t=0}: f_{0}=f\right\}}{\left\{\left.\frac{d}{d t} f \circ \varphi_{t}\right|_{t=0}\right\}}
$$

is the self-same Jacobian algebra, and indeed the Jacobian ideal itself is the extended tangent space for right-equivalence. The analogue of Theorem 3.37 shows that one can construct a versal
deformation of $f$ (versal for right-equivalence, that is) by taking $g_{1}, \ldots, g_{\mu} \in \mathcal{O}_{\mathbb{C}^{n+1}, 0}$ whose images in the Jacobian algebra span it as vector space, and defining

$$
F\left(x, u_{1}, \ldots, u_{\mu}\right)=f(x)+\sum_{j} u_{j} g_{j} .
$$

The Milnor fibration extends to a fibration over the complement of the discriminant $\Delta$ in the basespace $S=\mathbb{C}^{\mu}$; taking its associated cohomology bundle we obtain a holomorphic vector bundle of rank $\mu$ over the $\mu$-dimensional space $S$. It is equipped with a canonical flat connection, the Gauss-Manin connection.

The objective now is to show that many of these same ingredients can be found in the theory of singularities of mappings.

### 4.2 Image Milnor Number and Discriminant Milnor Number

We have already seen, in Example 3.39, that the real image of each codimension 1 germ $f$ of mappings from surfaces to 3 -space grows a 2 -dimensional homotopy-sphere when $f$ is suitably perturbed.

Proposition 4.1. (1) Suppose that $f:\left(\mathbb{C}^{n}, S\right) \rightarrow\left(\mathbb{C}^{n+1}, 0\right)$ is a map-germ of finite codimension. Then the image of a stable perturbation of $f$ has the homotopy type of a wedge of $n$-spheres.
(2 Suppose that $f:\left(\mathbb{C}^{n}, S\right) \rightarrow\left(\mathbb{C}^{n+1}, 0\right)$ is a map-germ of finite codimension, with $n \geq p$. Then the discriminant ( $=$ set of critical values) of a stable pertubation of $f$ has the homotopy-type of a wedge of $(p-1)$-spheres.

Terminology The number of spheres in the wedge is called the image Milnor number, $\mu_{I}$, in case (1), and the discriminant Milnor number, $\mu_{\Delta}$, in case (2).

Proof of 4.1 Both statements are consequences of a fibration theorem of Lê Dung Trang ([47]), that says, in effect, that if $\left(X, x_{0}\right)$ is a $p$-dimensional complete intersection singularity and $\pi:\left(X, x_{0}\right) \rightarrow(\mathbb{C}, 0)$ is a function with isolated singularity, in a suitable sense, then the analogue of the Milnor fibre of $\pi$ (i.e. the intersection of a non-zero level set with a Milnor ball around $x_{0}$ ) has the homotopy-type of a wedge of spheres of dimension $p-1$. To apply this theorem here, we take, as $X$, the germ of the image in case (1), or discriminant, in case (2), of a 1-parameter stabilisation of $f$ : that is, an unfolding $F:\left(\mathbb{C}^{n} \times \mathbb{C}, S \times\{0\}\right) \rightarrow\left(\mathbb{C}^{p} \times \mathbb{C}, 0\right)$ with $F(x, u)=(\tilde{f}(x, u), u)=\left(f_{u}(x), u\right)$ such that $f_{u}$ is stable for $u \neq 0$. Then $(X, 0)$ is a hypersurface singularity, and thus a complete intersection. We take, as $\pi$, the projection to the parameter space. Thus $\pi^{-1}(u)$ is the image (or discriminant) of $f_{u}$. The fact that $\pi$ has isolated singularity is a consequence of the fact that $f_{u}$ is stable for $u \neq 0$. For this implies that the unfolding is trivial away from $u=0$, so that the vector field $\partial / \partial u$ in the target of $\pi$ lifts to a vector field tangent to $X$.

## Discriminant of stable perturbation of the bi-germ

$$
\left\{\begin{array}{c}
(u, v, w) \mapsto\left(u, v, w^{3}-u w\right) \\
(x, y, z) \mapsto\left(x, y^{3}+x y, z\right)
\end{array}\right.
$$



Siersma proves in [45] that the number of spheres in the wedge is counted by the sum of the Milnor numbers of the isolated critical points of the defining equation $g$ of the image/discriminant which move off the image/discriminant as $f$ (and with it $g$ ) is deformed. The proof can be understood as follows. Let $g_{u}: B_{\varepsilon} \rightarrow \mathbb{C}$ be a reduced defining equation for the image/discriminant of $f_{u}$, varying analytically with $u$ for $u \in(\mathbb{C}, 0)$. We apply Morse theory. Up to homotopy, the space $B_{\varepsilon}$ is obtained from $g_{u}^{-1}(0)$ by progressively thickening it: considering

$$
\left|g_{u}\right|^{-1}([0, \eta])
$$

and increasing $\eta$. Away from critical points of $\left|g_{u}\right|$, this thickening does not change the homotopy type. Changes in homotopy-type occur only when $\eta$ passes through a critical value of $\left|g_{u}\right|$. The critical points of $\left|g_{u}\right|$ off $g_{u}^{-1}(0)$ are the same as those of $g_{u}$, and each has index equal to the ambient dimension, because of the complex structure. Thus, the contractible space $B_{\varepsilon}$ is obtained from $g_{u}^{-1}(0)$ by gluing in cells of dimension $p$. It follows by a standard Mayer-Vietoris type argument that $g^{-1}(0)$ is homotopy-equivalent to the wedge of the boundaries of these cells. We can assume that $g_{u}$ has only non-degenerate critical points off $g_{u}^{-1}(0)$; so the number of cells is the sum of their Milnor numbers.
This counting procedure is esential for the proofs of the following theorems.
Theorem 4.2. ([9]) Let $f:\left(\mathbb{C}^{n}, S\right) \rightarrow\left(\mathbb{C}^{p}, 0\right)$ be a map-germ of finite codimension, with $n \geq p$ and $(n, p)$ nice dimensions. Then

$$
\mu_{\Delta}(f) \geq \mathscr{A}_{e}-\operatorname{codim}(f)
$$

with equality if $f$ is weighted homogeneous.
Theorem 4.3. Let $f:\left(\mathbb{C}^{n}, S\right) \rightarrow\left(\mathbb{C}^{n+1}, 0\right)(n=1$ or 2$)$ have finite codimension. Then

$$
\begin{equation*}
\text { (1) } \mu_{I}(f) \geq \mathscr{A}_{e}-\operatorname{codim}(f) \quad \text { (2)Equality holds if } f \text { is weighted homogeneous. } \tag{4.1}
\end{equation*}
$$

Theorem 4.3 was proved for $n=2$ by de Jong and van Straten in [10]; another proof, also inspired by de Jong and van Straten, was given in [37], and an analogous proof for the case $n=1$ was given in [38].

A number of examples $([6],[21],[20],[41])$ of map-germs $\left(\mathbb{C}^{n}, 0\right) \rightarrow\left(\mathbb{C}^{n+1}, 0\right)$ for $n \geq 3$ support the "Mond conjecture" that (4.1) should hold for all $n$ for which $(n, n+1)$ are nice dimensions, but it remains unproven. Part of the difficulty in proving the conjecture lies in the fact that we do not have an effective method for computing image Milnor numbers. The best we can do here involves the image-computing spectral sequence (see [16], [17], [19]), and this only yields an answer when $f$ has corank 1.

### 4.3 Sections of stable images and discriminants

In contrast, we do have a method for computing discriminant Milnor numbers.
To explain it we begin by simplifying our initial description of $T^{1}(f)$, using an idea of Damon. If $F: X \rightarrow Y$ and $i: Z \rightarrow Y$ are two maps, the fibre product of $X$ and $Z$ over $Y$, denoted by $X \times_{Y} Z$, is the space

$$
X \times_{Y} Z=\{(x, z) \in X \times Z: F(x)=i(z) .\}
$$

A fibre product diagram is the commutative diagram

which results, where $\pi_{Z}$ and $\pi_{X}$ are the restrictions to $X \times_{Y} Z$ of the projections $X \times Z \rightarrow Z$ and $X \times Z \rightarrow X$. If $X, Y$ and $Z$ are smooth spaces and $i \pitchfork F$ then $X \times_{Y} Z$ is smooth also. In this case we say that $f$ is the transverse pull-back of $F$ by $i$, and write $f=i^{*}(F)$. The transversality of $i$ to $F$ guarantees that $X \times_{Y} Z$ is smooth, but there is no canonical choice of coordinate system on $X \times_{Y} Z$, so the map $i^{*}(F)$ is really defined only up to right-equivalence.

Exercise 4.4. If $f$ is obtained by transverse pull-back from $F$ then

1. the set of critical points of $f$ is the preimage by $\pi_{X}$ of the set of critical points of $F$;
2. the set of critical values of $f$ is the preimage by $i$ of the set of critical values of $F$;
3. the local algebras $Q(f)$ and $Q(F)$ are isomorphic.

Example 4.5. Take $X=\mathbb{C}^{2}, Y=\mathbb{C}^{3}, Z=\mathbb{C}^{3}$, and let $F(u, v)=\left(x_{1}, x_{2}^{2}, x_{1} x_{2}\right)$ and $i\left(z_{1}, z_{2}, z_{3}\right)=$ $\left(p\left(z_{1}, z_{2}\right), z_{2}, z_{3}\right)$. Then $i \pitchfork F$ and

$$
X \times_{Y} Z=\left\{\left(x_{1}, x_{2}, z_{1}, z_{2}, z_{3}\right): x_{1}=p\left(z_{1}, z_{2}\right), x_{2}^{2}=z_{2}, x_{1} x_{2}=z_{3 .}\right\}
$$

The three equations defining $X \times_{Y} Z$ allow us to dispense with the coordinates $x_{1}, z_{2}$ and $z_{3}$ and retain $z_{1}, x_{2}$ as coordinates on $X \times_{Y} Z$. The maps $\pi_{X}$ and $\pi_{Z}$ are then given by

$$
\begin{aligned}
& \pi_{Z}\left(z_{1}, x_{2}\right)=\left(z_{1}, x_{2}^{2}, x_{2} p\left(z_{1}, x_{2}^{2}\right)\right. \\
& \pi_{X}\left(z_{1}, x_{2}\right)=\left(p\left(z_{1}, x_{2}^{2}\right), x_{2}\right)
\end{aligned}
$$

Exercise 4.6. 1. Let $f$ be the germ of type $H_{2}$ given by $(x, y) \mapsto\left(x, y^{3}, x y+y^{5}\right)$ and let $F(a, b, c, y)=\left(a, b, c, y^{3}+a y, b y^{2}+c y\right)$. Find $i: \mathbb{C}^{3} \rightarrow \mathbb{C}^{5}$ such that $i^{*}(F) \simeq_{\mathcal{A}} f$.
2. Let $F(u, v, y)=\left(u, v, y^{4}+u y^{2}+v y\right)$. Find $i:\left(\mathbb{C}^{2}, 0\right) \rightarrow\left(\mathbb{C}^{3}, 0\right)$ such that $i^{*}(F) \simeq_{\mathcal{A}} f$ where $f(x, y)=\left(x, x y+y^{4}\right)$.
3. Let $f(x, y)=\left(x, y^{3}+x^{k} y\right)$. Find a stable germ $F$ and a germ $i$ such that $f \simeq_{\mathcal{A}} i^{*}(F)$.
4. Find a stable bi-germ $F:\left(\mathbb{C},\left\{0,0^{\prime}\right\}\right) \rightarrow\left(\mathbb{C}^{2}, 0\right)$ and $i:\left(\mathbb{C}^{2}, 0\right) \rightarrow\left(\mathbb{C}^{2}, 0\right)$ such that $i^{*} F$ is $\mathcal{A}$-equivalent to

$$
\left\{\begin{array}{rll}
s & \mapsto & \left(s, s^{2}\right) \\
t & \mapsto & \left(t,-t^{2}\right)
\end{array}\right.
$$

Every map-germ $f:\left(\mathbb{C}^{n}, S\right) \rightarrow\left(\mathbb{C}^{p}, 0\right)$ of finite singularity type can be obtained by transverse pull-back from a stable map-germ: simply construct a stable unfolding $F:\left(\mathbb{C}^{n} \times \mathbb{C}^{d}, 0\right) \rightarrow\left(\mathbb{C}^{p} \times \mathbb{C}^{d}, 0\right)$ (along the lines described in Remark 3.22), and then recover $f$ from $F$ by the map $i:\left(\mathbb{C}^{p}, 0\right) \rightarrow\left(\mathbb{C}^{p} \times \mathbb{C}^{d}, 0\right)$ given by $i(y)=(y, 0)$. This may not be the simplest such procedure but it always works.

Definition 4.7. If $F:\left(X, x_{0}\right) \rightarrow\left(Y, y_{0}\right)$ is any map-germ, the isosingular locus of $F$ is the set-germ

$$
\mathscr{I}_{F}:=\left\{y \in\left(Y, y_{0}\right): f:\left(X, F^{-1}(y) \cap \sum_{F} \rightarrow(Y, y) \text { is } \mathscr{A} \text {-equivalent to } F .\right\}\right.
$$

Just as the domain $X \times_{Y} Z$ of $f$ is smooth if and only if $i \pitchfork F$, the map $f$ is stable if and only if $i \pitchfork \mathscr{I}_{F}$. This suggests that the instability of $f$ should be reflected in the failure of $i$ to be transverse to $\mathscr{I}_{F}$. A theorem of Damon ( 4.13 below) makes this precise. We need

Definition 4.8. (1) If $D \subset Y$ is an analytic subvariety, $\operatorname{Der}(-\log D)$ is the $\mathcal{O}_{Y}$-module (sheaf) of germs of vector fields on $Y$ tangent to $D$ at its smooth points.
(2) If $D$ is a divisor (hypersurface) in $Y$, we say $D$ is a free divisor if $\operatorname{Der}(-\log D)$ is a locally free $\mathcal{O}_{Y}$-module.

It is easy to show that if $D$ is the variety of zeros of an ideal $I$ then

$$
\operatorname{Der}(-\log D)=\left\{\chi \in \theta_{Y}: \chi \cdot g \in I \text { for all } h \in I\right\}
$$

and in particular if $D$ is a hypersurface with equation $h$ then

$$
\operatorname{Der}(-\log D)=\left\{\chi \in \theta_{Y}: \chi \cdot h=\alpha h \text { for some } \alpha \in \mathcal{O}_{Y}\right\}
$$

Let $F$ be a map-germ of finite $\mathscr{A}_{e}$-codimension, and let $\Delta(F)$ be its discriminant.
Proposition 4.9. $T_{y_{0}} \mathscr{I}_{F}=\left\{\chi\left(y_{0}\right): \chi \in \operatorname{Der}(-\log \Delta(F))_{y_{0}}\right\}$.
To prove this we need
Lemma 4.10. If $\chi \in \operatorname{Der}(-\log \Delta(F))$ then it can be lifted to a vector field $\tilde{\chi}$ on $X-$ that is, there exists $\tilde{\chi} \in \theta_{X}$ such that

$$
t F(\tilde{\chi})=\omega F(\chi)
$$

Proof of Proposition 4.9 For a proof of the lemma, see e.g. [24, 6.14]. Using the lemma, suppose $\chi \in \operatorname{Der}(-\log \Delta(F))$ and let $\tilde{\chi}$ be a lift. If $\tilde{\varphi}_{t}$ and $\varphi_{t}$ are the integral flows of $\tilde{\chi}$ and $\chi$ then by Corollary 3.5, we have $F \circ \tilde{\varphi}_{t}=\varphi_{t} \circ F$. This shows that any orbit of $\chi$ is tangent to the isosingular locus of $F$, and shows inclusion of right hand side in left hand side in 4.9. The converse is (almost) clear from the definition of isosingular locus; an argument of Ephraim ([13]) makes this precise.

The vector space on the right is known as the logarithmic tangent space to $\Delta(F)$ at $y_{0}$; we denote it by $T_{y_{0}}^{\log } \Delta(F)$.

Proposition 4.11. If $F:\left(\mathbb{C}^{n}, S\right) \rightarrow\left(\mathbb{C}^{p}, Y\right)(n \geq p)$ is stable then $\Delta(F)$ is a free divisor.

Proof Looijenga's book [24, 6.13].
Given a diagram

we measure the failure of transversality of $i$ to $\mathscr{I}_{F}$ by the module

$$
\begin{equation*}
\frac{\theta(i)}{t i\left(\theta_{Y}\right)+i^{*}(\operatorname{Der}(-\log \Delta(F))}, \tag{4.3}
\end{equation*}
$$

which is denoted by $T_{\mathscr{K}_{\Delta(F)}}^{1} i$. We say $i$ is "logarithmically transverse to $\Delta(F)$ " at $z$ if

$$
d_{z} i\left(T_{z} Z\right)+T_{i(z)} \Delta(F)=T_{i z} Y
$$

Proposition 4.12. Let $D \subset Y$ be a hypersurface and $i: Z \rightarrow Y$ a map. Then $i$ is logarithmically transverse to $D$ at $z$ if and only if $T_{\mathscr{K}_{D}}^{1} i=0$.

Proof Nakayama's Lemma
Theorem 4.13. (J.N.Damon,[8]) If $f$ is obtained from the stable map $F$ by transverse pull back by $i$ then

$$
T^{1}(f) \simeq T_{\mathscr{K}_{\Delta(F)}}^{1} i
$$

A simpler proof than Damon's original one can be found in [39, Section 8]. The module in the denominator of (4.3) is in fact the (extended) tangent space to the orbit of $i$ under a variant of contact equivalence introduced by Damon in [7] and called $\mathcal{K}_{V}$-equivalence, though we will not make use of this here. In Damon's original proof of 4.13 in [8] he showed that if $i_{t}$ is a deformation of $i$ then the family $i_{t}^{*}(F)$ is $\mathcal{A}$-trivial if and only if $i_{t}$ is $\mathcal{K}_{\Delta(F)}$-trivial.

Definition 4.14. Let $f, g:\left(Z, z_{0}\right) \rightarrow\left(Y, y_{0}\right)$ and let $\left(V, y_{0}\right) \subset\left(Y, y_{0}\right)$. We say that $f$ is $\mathcal{K}_{V^{-}}$ equivalent to $g$ if there exists diffeomorphisms $\Phi:\left(Z \times Y,\left(z_{0}, y_{0}\right)\right) \rightarrow\left(Z \times Y,\left(z_{0}, y_{0}\right)\right)$ and $\varphi$ : $\left(Z, z_{0}\right) \rightarrow\left(Z, z_{0}\right)$ such that

1. $\Phi$ lifts $\varphi$, i.e. $\pi_{Z} \circ \Phi=\varphi \circ \pi_{Z}$;
2. $\Phi(Z \times V)=Z \times V$,
3. $\Phi(\operatorname{graph}(f))=\operatorname{graph}(g)$.

In the usual version of contact $(\mathcal{K})$ equivalence, $V=\left\{y_{0}\right\}$.
The advantage of the expression (4.3) over the expression (3.22) is that in (4.3) all the objects are finite modules over $\mathcal{O}_{Z}$, whereas the first summand in the denominator in (3.22) is an $\mathcal{O}_{\mathbb{C}^{n}, S^{-}}$ module while the second is only an $\mathcal{O}_{\mathbb{C}^{p}, 0}$ module. This makes (4.3) algebraically much simpler to work with.

Let $h$ be an equation of $\Delta(F)$, and define $\operatorname{Der}(-\log h)$ to be the $\mathcal{O}_{Y}$-module of germs of vector fields which annihilate $h$; that is, which are tangent not only to $\Delta(F)=h^{-1}(0)$, but to all level sets of $h$. Clearly $\operatorname{Der}(-\log h)$ is a submodule of $\operatorname{Der}(-\log D)$, but it depends on the choice of equation $h$, and is not determined by $D$ alone.

Theorem 4.15. ([9]) If $f:\left(\mathbb{C}^{n}, S\right) \rightarrow\left(\mathbb{C}^{p}, 0\right)$, with $n \geq p$ and $(n, p)$ nice dimensions, and $f$ is obtained from the stable map-germ $F$ by transverse pull back by $i$, then

$$
\begin{equation*}
\mu_{\Delta}(f)=\operatorname{dim}_{\mathbb{C}} \frac{\theta(i)}{t i\left(\theta_{\mathbb{C}^{p}, 0}\right)+i^{*}(\operatorname{Der}(-\log h))} \tag{4.4}
\end{equation*}
$$

The proof of this result uses conservation of multiplicity, and depends in an essential way on the fact that $\Delta(F)$ is a free divisor. The inequality in Theorem 4.2 follows immediately from 4.15 and 4.13 .

### 4.4 Open questions

1. The "Mond conjecture" asserts that if $f:\left(\mathbb{C}^{n}, S\right) \rightarrow\left(\mathbb{C}^{n+1}, 0\right)$ is a map-germ of finite codimension, and $(n, n+1)$ are in Mather's nice dimensions, then $\mu_{I}(f) \geq \mathcal{A}_{e}-\operatorname{codim}(f)$, with equality if $f$ is weighted homogeneous. It is proved for $n=1$ and $n=2$, and supported by many examples.
2. A famous theorem of Lê and Ramanujan states that a $\mu$-constant family of isolated hypersurface singularities is topologically trivial, provided the ambient dimension is not 3 . It is unknown whether this holds also in dimension 3. Do the image and discriminant Milnor numbers have an equally crucial role in determining the topology?
3. A stable perturbation of a finitely determined real map-germ $\left(\mathbb{R}^{n}, S\right) \rightarrow\left(\mathbb{R}^{n+1}, 0\right)$ is maximal if it exhibits all of the 0-dimensional stable singularities present in its complexification. It is a good real perturbation if the real image has $n$ 'th homology of rank $\mu_{I}(f)$ (so that inclusion of real image in complex image induces an isomorphism on $H_{n}$ ). Is it true that every good real perturbation is maximal? This is the case in all known examples. The same question is also open, concerning maps $\left(\mathbb{R}^{n}, S\right) \rightarrow\left(\mathbb{R}^{p}, 0\right)$ with $n \geq p$, with "discriminant" replacing "image" and $\mu_{\Delta}$ replacing $\mu_{I}$.

## 5 Multiple Points

The multiple point spaces of a map-germ $\left(\mathbb{C}^{n}, 0\right) \rightarrow\left(\mathbb{C}^{p}, 0\right)$ with $n<p$ play an important rôle in the study of its geometry, as well as the topology of the image of a stable perturbation ([37], [26], [16], [17]).

The $k$ 'th source multiple point space $D^{k}$ of a finite proper map between topological spaces is the closure of the set of $k$-tuples of pairwise distinct points having the same image under the map. The $k$ 'th target multiple point space $M_{k}(f)$ is the closure in the image of the set of points having $k$ or more distinct preimages. When $f: X \rightarrow Y$ is a finite analytic map of complex manifolds, the space $M_{k}(f)$ has a natural analytic structure as the subspace of $Y$ defined by the $(k-1)$ 'st Fitting ideal $\operatorname{Fitt}_{k-1}\left(f_{*} \mathcal{O}_{X}\right)$ of the pushforward $f_{*} \mathcal{O}_{X}$ (see [46], [40], [23]). This structure is particularly good when $X$ is Cohen-Macaulay, $Y$ is smooth and $\operatorname{dim} Y=\operatorname{dim} X+1$. One might hope for an analogous formula giving equations for $D^{k}(f)$ in $X^{k}$, in terms of $f$ itself. No such formula is known in general, though for $k=2$ the ideal defined, in terms of local coordinates $x_{1}, \ldots, x_{n}$ and $y_{1}, \ldots, y_{p}$ on $X$ and $Y$, by

$$
\begin{equation*}
\mathcal{I}_{2}:=(f \times f)^{*} I_{\Delta_{p}}+\operatorname{Fitt}_{0}\left(I_{\Delta_{n}} /(f \times f)^{*} I_{\Delta_{p}}\right) \tag{5.1}
\end{equation*}
$$

where $I_{\Delta_{n}}$ and $I_{\Delta_{p}}$ are the ideal sheaves defining the diagonals $\Delta_{n}$ in $\mathbb{C}^{n} \times \mathbb{C}^{n}$ and $\Delta_{p}$ in $\mathbb{C}^{p} \times \mathbb{C}^{p}$, gives $D^{2}(f)$ a scheme structure with many desirable qualities: if $f$ is dimensionally correct - that is, if $D^{2}(f)$ has the expected dimension, $2 n-p$, then $D^{2}(f)$ is Cohen Macaulay. If moreover $f$ is finitely determined (for left-right equivalence), or, equivalently, has isolated instability, then provided its dimension is greater than $0, \mathcal{I}_{2}$ is radical.

If the corank of $f$ (the dimension of $\operatorname{Ker} d f_{0}$ ), is equal to 1 , much more is possible. An explicit list of generators for the ideal defining $D^{k}(f)$ in $\left(\mathbb{C}^{n}\right)^{k}$ is given in [26], where it is shown that a finite corank 1 map-germ $f:\left(\mathbb{C}^{n}, 0\right) \rightarrow\left(\mathbb{C}^{p}, 0\right)$ is stable if and only if each $D^{k}(f)$ is smooth of dimension $p-k(p-n)$, or empty, for all $k \geq 2$. Moreover, it is finitely $\mathcal{A}$-determined if and only if $D^{k}$ is an ICIS of dimension $p-k(p-n)$ or empty for those $k$ with $p-k(p-n) \geq 0$, and $D^{k}$ consists at most of only the origin if $p-k(p-n)<0$ (see, e.g., [25], [16] for other results).

We will say that $f$ is dimensionally correct if for each $k, D^{k}(f)$ satisfies these dimensional requirements, including the requirement that when $p-k(p-n)<0, D^{k}(f)$ consists at most of the origin.

### 5.1 Multiple point spaces.

Given a map $f: X \rightarrow Y$, we set

$$
\begin{equation*}
{ }^{o} D^{k}(f)=\left\{\left(x_{1}, \ldots, x_{k}\right) \in X^{k} \mid f\left(x_{1}\right)=\cdots=f\left(x_{k}\right), x_{i} \neq x_{j} \text { if } i \neq j\right\} \tag{5.2}
\end{equation*}
$$

and define the $k$ 'th source multiple point space of $f, D^{k}(f)$, by

$$
\begin{equation*}
D^{k}(f)=\text { closure }{ }^{o} D^{k}(f) \tag{5.3}
\end{equation*}
$$

(where the closure in taken in $X^{k}$ ) provided ${ }^{o} D^{k}(f)$ is not empty. We extend this definition to germs of maps by taking the limit over representatives; if $f \in \mathcal{E}_{n, p}^{0}$ is finite, the local conical structure guarantees that we obtain in this way a well defined germ at $\mathbf{0} \in\left(\mathbb{C}^{n}\right)^{k}$. We give $D^{k}(f)$ an analytic structure as follows. First, choose a stable unfolding $F: X \times \mathbb{C}^{d} \rightarrow Y \times \mathbb{C}^{d}$ and give $D^{k}(F)$ its
reduced structure. Because $F$ is an unfolding, $D^{k}(F)$ embeds naturally in $X^{k} \times \mathbb{C}^{d}$, with defining ideal $\mathcal{I}_{k}(F)$. Define

$$
\mathcal{I}_{k}(f)=\left.I_{k}(F)\right|_{\mathbf{u}=0} .
$$

It is straightforward to check that this is independent of the choice of stable unfolding, and is compatible with unfolding in the sense that for any germ of unfolding $F:\left(\mathbb{C}^{n} \times \mathbb{C}^{d}, 0\right) \rightarrow\left(\mathbb{C}^{p} \times \mathbb{C}^{d}, 0\right)$ of $f$, the diagram

in which the vertical arrows are projections to the base and the horizontal arrows are inclusions, is a fibre square.

This definition of $\mathcal{I}_{k}(f)$ is canonical, but gives no hint as to how to $\mathcal{I}_{k}(f)$ is to be calculated. But suppose that $f:\left(\mathbb{C}^{n}, 0\right) \rightarrow\left(\mathbb{C}^{p}, 0\right)$ has corank 1 . Then with respect to suitable coordinates it can be written in the form

$$
\begin{equation*}
f(\mathbf{x}, y)=\left(\mathbf{x}, f_{n}(\mathbf{x}, y), \ldots, f_{p}(\mathbf{x}, y)\right) \tag{5.5}
\end{equation*}
$$

where $(\mathbf{x}, y)$ are suitable coordinates on $\mathbb{C}^{n}$. That is, we write $f$ explicitly as an unfolding of a map-germ in the single variable $y$. Now any $k$ points $\left(\mathbf{x}_{1}, y_{1}\right), \ldots,\left(\mathbf{x}_{k}, y_{k}\right)$ sharing the same image must have equal $\mathbf{x}$ coordinates, and so $D^{k}(f)$ embeds naturally in $\mathbb{C}^{n-1} \times \mathbb{C}^{k}$. We take coordinates $\mathbf{x}, y_{1}, \ldots, y_{k}$ on $\mathbb{C}^{n-1} \times \mathbb{C}^{k}$ and look for equations defining $D^{k}(f)$ in $\mathbb{C}^{n-1} \times \mathbb{C}^{k}$.

The following analysis will be applied to each of the component functions $f_{j}, j=n \ldots, p$ of $f$. To spare notation for the moment, let $h$ be any function of $x, y_{1}, \ldots, y_{k}$. The map

$$
\begin{equation*}
\left(x, y_{1}, \ldots, y_{k}\right) \mapsto\left(h\left(x, y_{1}\right), \ldots, h\left(x, y_{p}\right)\right) \tag{5.6}
\end{equation*}
$$

is equivariant with respect to the symmetric group actions on the source permuting the $y_{i}$ and on the target permuting the $f_{j}\left(x, y_{i}\right)$. The set $\mathcal{E}^{S_{k}}$ of equivariant maps $\mathbb{C}^{n-1+k} \rightarrow \mathbb{C}^{k}$ is a module over the ring $\mathcal{O}^{S_{k}}$ of invariant functions on the source, generated (although we will not need this fact) by the gradient vectors of the generators of $\mathcal{O}^{S_{k}}([43])$. The ring $\mathcal{O}^{S_{k}}$ is generated over $\mathcal{O}_{\mathbb{C}^{n-1}, 0}$ by the sums of powers $\rho_{1}=y_{1}+\cdots+y_{k}, \ldots, \rho_{k}=y_{1}^{k}+\cdots+y_{k}^{k}$, and so every equivariant mapping can be written as linear combination, over $\mathcal{O}^{S_{k}}$, of the maps

$$
\begin{align*}
m_{1}\left(y_{1}, \ldots, y_{k}\right) & =(1, \ldots, 1) \\
m_{2}\left(y_{1}, \ldots, y_{k}\right) & =\left(y_{1}, \ldots, y_{k}\right)  \tag{5.7}\\
\cdots & \cdots \\
m_{k-1}\left(y_{1}, \ldots, y_{k}\right) & =\left(y_{1}^{k-1}, \ldots, y_{k}^{k-1}\right)
\end{align*}
$$

Thus there exist invariant functions $\alpha_{0}^{k}, \alpha_{1}^{k}, \ldots, \alpha_{k-1}^{k}$ such that

$$
\left(\begin{array}{c}
h\left(x, y_{1}\right)  \tag{5.8}\\
\vdots \\
h\left(x, y_{k}\right)
\end{array}\right)=\alpha_{0}^{k}\left(\begin{array}{c}
1 \\
\vdots \\
1
\end{array}\right)+\alpha_{1}^{k}\left(\begin{array}{c}
y_{1} \\
\vdots \\
y_{k}
\end{array}\right)+\cdots+\alpha_{k-1}^{k}\left(\begin{array}{c}
y_{1}^{k-1} \\
\vdots \\
y_{k}^{k-1}
\end{array}\right)
$$

Solving for the $\alpha_{i}^{k}$ by by Cramer's rule gives

$$
\left.\alpha_{\ell}^{k}\left(x, y_{1}, \ldots, y_{k}\right)=\frac{\left|\begin{array}{cccccccc}
1 & y_{1} & \cdots & y_{1}^{\ell-1} & h\left(x, y_{1}\right) & y_{1}^{\ell+1} & \cdots & y_{1}^{k-1}  \tag{5.9}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & y_{k} & \cdots & y_{k}^{\ell-1} & h\left(x, y_{k}\right) & y_{k}^{\ell+1} & \cdots & y_{k}^{k-1}
\end{array}\right|}{} \begin{array}{|ccccc}
1 & y_{1} & \cdots & y_{1}^{k-1} \\
\vdots & \vdots & \vdots & \vdots \\
1 & y_{k} & \cdots & y_{k}^{k-1}
\end{array} \right\rvert\, \text { }
$$

In fact we do not need Poenaru's statement, referred to above, to see that the $\alpha_{\ell}^{k}$ are regular (analytic): the numerator in (5.9) vanishes whenever $y_{i}=y_{\ell}$ for any $i, \ell$, and thus is divisible in $\mathcal{O}$ by $\prod_{i<\ell}\left(y_{i}-y_{\ell}\right)$, i.e. by the Vandermonde determinant, which is the denominator in (5.9). In other words the system of equations (5.8) has analytic solutions. As can be seen from (5.9), they are $S_{k}$-invariant . They are also unique, since the Vandermonde determinant vanishes only along a hypersurface.

Let $I_{k}(h)$ be the ideal generated by the $\alpha_{\ell}^{k}$ for $\ell=1, \ldots, k-1$.
Remark 5.1. The ideal $I_{k}(h)$ is also generated over $\mathcal{O}_{\mathbb{C}^{n-1} \times \mathbb{C}^{k}, 0}$ by the $k-1$ functions $R_{i}(h)$, for $i=2, \ldots, k$, which are defined iteratively by

$$
\begin{align*}
R_{2}(h)\left(\mathbf{x}, y_{1}, y_{2}\right) & =\frac{f_{j}\left(\mathbf{x}, y_{2}\right)-f_{j}\left(\mathbf{x}, y_{1}\right)}{y_{2}-y_{1}} \text { and } \\
R_{i}(h)\left(\mathbf{x}, y_{1}, \ldots, y_{i+1}\right) & =\frac{R_{i-1}(h)\left(\mathbf{x}, y_{1}, \ldots, y_{i-1}, y_{i+1}\right)-R_{i-1}(h)\left(\mathbf{x}, y_{1}, \ldots, y_{i-1}, y_{i}\right)}{y_{i+1}-y_{i}} . \tag{5.10}
\end{align*}
$$

Theorem 5.2. ([26]) If $f$ as in (5.5) is dimensionally correct then $\mathcal{I}_{k}(f)=I_{k}\left(f_{n+1}\right)+\cdots+I_{k}\left(f_{p}\right)$.
Thus we have $(k-1)(p-n+1)$ explicit equations for $D^{k}(f)$.
Exercise 5.3. 1. Find equations for $D^{2}(f)$ and $D^{3}(f)$ when $f$ is the map-germ given by
(a) $f\left(x_{1}, x_{2}, x_{3}, y\right)=\left(x_{1}, x_{2}, x_{3}, y^{3}+x_{1} y, x_{2} y^{2}+x_{3} y\right)$ (stable map-germ of type $\sum^{1,1,0}$ ).
(b) $f(x, y)=\left(x, y^{2}, y^{3}+x^{k+1} y\right)\left(\right.$ type $S_{k}$ in [36] - here $\left.D^{3}(f)=\emptyset\right)$
(c) $f(x, y)=\left(x, y^{3}, x y+y^{5}\right)\left(\right.$ type $H_{2}$ in [36])
(d) $f(x, y)=\left(x, y^{3}, x y+y^{3 k-1}\right)\left(\right.$ type $H_{k}$ in [36]).
2. In $1(\mathrm{a})$, check that $D^{k}(f)$ is smooth whenever non-empty.
3. For $1(\mathrm{~b}), 1(\mathrm{c})$ and $1(\mathrm{~d})$, check that $D^{2}(f)$ has isolated singularity.
4. For $1(\mathrm{c})$ and $1(\mathrm{~d})$, check that $D^{3}(f)$ is zero-dimensional. What is the complex vector space dimension of $\mathcal{O}_{\mathbb{C} \times \mathbb{C}^{3}, 0} / \mathcal{I}_{3}(f)$ in these two cases (your answer should be divisible by 6 )?
5. Suppose that $f:\left(\mathbb{C}^{n}, 0\right) \rightarrow\left(\mathbb{C}^{n+1}, 0\right)$ has corank 1 and is finitely determined. Show that
(a) $\operatorname{dim}_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^{n-1} \times \mathbb{C}^{n+1}, 0} / \mathcal{I}_{n+1}(f)$ is divisible by $(n+1)$ !.
(b) and use Theorem 2.24 to show that if $f_{t}$ is a stable perturbation of $f$, then the image of $f_{t}$ contains

$$
\frac{1}{(n+1)!}\left(\operatorname{dim}_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^{n-1} \times \mathbb{C}^{n+1}, 0} / \mathcal{I}_{n+1}(f)\right)
$$

ordinary $(n+1)$-tuple points (at which it is locally isomorphic to the union of the $n+1$ coordinate hyperplanes in $\left(\mathbb{C}^{n+1}, 0\right)$ ).

Theorem 5.4. ([26]) Let $f:\left(\mathbb{C}^{n}, S\right) \rightarrow\left(\mathbb{C}^{p}, 0\right)$, with $n<p$, have corank 1 .

1. $f$ is stable if and only if for each $k$ with $\geq(k-1) p, D^{k}(f)$ is smooth of dimension $k n-(k-1) p$, and $D^{k}(f)=\emptyset$ if $k n<(k-1) p$.
2. $f$ is finitely determined if and only if for each $k$ with $k n \geq(k-1) p, D^{k}(f)$ is an isolated complete intersection singularity of dimension $k n-(k-1) p$, and $D^{k}(f)=\{0\}$ or $\emptyset$ if $k n<$ $(k-1) p$.

As a result of the two parts of Theorem 5.4, it follows that when $f_{t}$ is a stable perturbation of a finitely determined corank 1 germ $f$, then $D^{k}\left(f_{t}\right)$ is a smoothing, and therefore a Milnor fibre, of the ICIS $D^{k}(f)$.

Exercise 5.5. Find the Milnor numbers of $D^{2}(f)$ and $D^{3}(f)$ for the map germs of type $S_{k}, H_{2}$ and $H_{k}$ in Exercise 5.3.

Now in (5.8) subtract the first row from each of the others. Omitting the first row in the resulting equation gives

$$
\left(\begin{array}{c}
h\left(x, y_{2}\right)-h\left(x, y_{1}\right)  \tag{5.11}\\
\vdots \\
h\left(x, y_{k}\right)-h\left(x, y_{1}\right)
\end{array}\right)=\left(\begin{array}{ccc}
\left(y_{2}-y_{1}\right) & \cdots & \left(y_{2}^{k-1}-y_{1}^{k-1}\right) \\
\vdots & \vdots & \vdots \\
\left(y_{k}-y_{1}\right) & \cdots & \left(y_{k}^{k-1}-y_{1}^{k-1}\right)
\end{array}\right)\left(\begin{array}{c}
\alpha_{1}^{k} \\
\vdots \\
\alpha_{k}^{k-1}
\end{array}\right)
$$

The determinant of the new matrix of coefficients is still $\operatorname{Vdm}\left(y_{1}, \ldots, y_{k}\right)$ (check this!) It follows that

$$
\begin{equation*}
I_{k}(h) \supseteq\left(h\left(x, y_{2}\right)-h\left(x, y_{1}\right), \ldots, h\left(x, y_{k}\right)-h\left(x, y_{1}\right)\right) \tag{5.12}
\end{equation*}
$$

and

$$
\begin{equation*}
y_{1}, \ldots, y_{k} \text { are pairwise distinct } \Longrightarrow I_{k}(h)=\left(h\left(x, y_{2}\right)-h\left(x, y_{1}\right), \ldots, h\left(x, y_{k}\right)-h\left(x, y_{1}\right)\right) \tag{5.13}
\end{equation*}
$$

By contrast, the restriction of $I_{k}(h)$ to the set $\left\{y_{1}=\cdots=y_{k}\right\}$ reduces to an ideal of partial derivatives.

For $k>\ell$ we define $D_{\ell}^{k}(f)$ to be the image in $D^{k}(f)$ of $D^{\ell}(f)$ under the composite $\pi_{\ell}^{\ell+1} \circ \cdots \circ \pi_{k-1}^{k}$. Then we have set-theoretic equalities $f^{(k)}\left(D^{k}(f)\right)=M_{k}(f)$ and $f^{-1} M_{k}(f)=D_{1}^{k}(f)$ for all $k \geq 1$.

### 5.2 Computing the homology of the image

Let $f: X \rightarrow Y$ be a finite map. For each $k \geq 2$ there are projections $D^{k}(f) \rightarrow D^{k-1}(f)$ defined by forgetting one of the copies of $X$. These give rise to maps on the vanishing homology of the Milnor fibres $D^{k}\left(f_{t}\right)$ when $f_{t}$ is a stable perturbation of $f$; there is thus a rather rich structure of homology groups and homomorphisms associated to a stable perturbation. It turns out that from this one
can obtain information about the homology of the image of the stable perturbation. The action of the symmetric group $S_{k}$ on $D^{k}(f)$ determines the gluing which takes place when the domain of $f_{t}$ is mapped to the image, and it is therefore no surprise that in the computation of the homology of the image, this action should play a rôle. In fact it is the alternating part of the homology which enters into the calculation of $H_{*}\left(\operatorname{image}\left(f_{t}\right)\right)$. This was first observed in [16] at the level of rational homology. For any map $f: X \rightarrow Y$, we define

$$
\operatorname{Alt}_{k} H_{q}\left(D^{k}(f) ; \mathbb{Q}\right)=\left\{[c] \in H_{q}\left(D^{k}(f) ; \mathbb{Q}\right): \sigma_{*}([c])=\operatorname{sign}(\sigma)[c] \text { for all } \sigma \in S_{k}\right\}
$$

and refer to it as the alternating part of $H_{q}\left(D^{k}(f) ; \mathbb{Q}\right)$. Later the construction was greatly clarified by Goryunov in [17], by the introduction of the alternating chain complex, which we now describe. The description here differs from Goryunov's only in that it uses singular homology in place of cellular homology.

### 5.3 The alternating chain complex

Let $D^{k}$ be any space on which the symmetric group $S_{k}$ acts, and let $C_{\ell}\left(D^{k}\right)$ be the usual free abelian group of singular $\ell$-chains in $D^{k}$. A chain $c \in C_{\ell}\left(D^{k}\right)$ is alternating if for each $\sigma \in S_{k}$, $\sigma_{\#}(c)=\operatorname{sign}(\sigma) c$. We denote the set of all alternating chains (with integer coefficients) on $D^{k}$ by $C_{\ell}^{\text {Alt }}\left(D^{k}\right)$. It is, evidently, a subgroup of $C_{\ell}\left(D^{k}\right)$, and therefore free abelian. The $C_{\ell}^{\text {Alt }}\left(D^{k}\right)$ form a complex under the usual boundary map; we call its homology the alternating homology of $D^{k}$ ), and denote it by $H_{*}^{\text {Alt }}\left(D^{k}\right)$.

## Proposition 5.6.

$$
H_{*}^{A l t}\left(D^{k}\right) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq A l t_{k} H_{q}\left(D^{k} ; \mathbb{Q}\right) .
$$

## Proof. Exercise

We will use this as a heuristic guide to later constructions. In particular, if $D^{k}=D^{k}\left(f_{t}\right)$, where $f_{t}$ is a stable perturbation of a corank 1 map-germ $\left(\mathbb{C}^{n}, 0\right) \rightarrow\left(\mathbb{C}^{p}, 0\right)$, then $D^{k}\left(f_{t}\right)$ is the Milnor fibre of an ICIS of dimension $p-k(p-n)$ provided $p-k(p-n) \geq 0$, and empty if $p-k(p-n)<0$; thus $H_{q}\left(D^{k}(f) ; \mathbb{Q}\right)=0$ unless $q=0$ or $q=p-k(p-n)$. Now if $p-k(p-n)>0, D^{k}(f)$ is connected and so $S_{k}$ acts trivially on $H_{0}\left(D^{k}(f) ; \mathbb{Q}\right)$, and it follows that $\operatorname{Alt}_{k} H_{0}\left(D^{k}(f) ; \mathbb{Q}\right)=0$. Thus

Proposition 5.7. If $f_{t}$ is a stable perturbation of a a corank 1 map-germ $\left(\mathbb{C}^{n}, 0\right) \rightarrow\left(\mathbb{C}^{n+1}, 0\right)$, then $\operatorname{Alt}_{k}\left(H_{q}\left(D^{k}\left(f_{t}\right) ; \mathbb{Q}\right)=0\right.$ if $q \neq p-k(p-n)$.

In other words, for all $k \operatorname{Alt}_{k} H_{*}\left(D^{k}\left(f_{t}\right) ; \mathbb{Q}\right)$ is concentrated in middle dimension.
Let us return to the situation of a map $f: X \rightarrow Y$, and let $D^{k}(f)$ be the usual multiple point spaces. Denote by $\pi^{k}$ the projection $D^{k}(f) \rightarrow D^{k-1}(f)$ defined by

$$
\pi^{k}\left(x_{1}, \ldots, x_{k}\right)=\left(x_{1}, \ldots, x_{k-1}\right)
$$

Proposition 5.8. $\pi_{\#}^{k}\left(C_{\ell}^{A l t}\left(D^{k}(f)\right) \subset C_{\ell}^{A l t}\left(D^{k-1}(f)\right)\right.$.
Proof. There is an obvious embedding $i: S_{k-1} \hookrightarrow S_{k}$ such that for $\sigma \in S_{k-1}$ then, as maps on $D^{k}(f)$,

$$
\sigma \circ \pi^{k}=\pi^{k} \circ i(\sigma) ;
$$

as a map $\{1, \ldots, k\} \rightarrow\{1, \ldots, k\}$,

$$
i(\sigma)(j)= \begin{cases}\sigma(j) & \text { if } j<k \\ k & \text { if } j=k\end{cases}
$$

The sign of $i(\sigma)$ is the same as the sign of $\sigma$; it follows that if $c \in C_{\ell}^{\text {Alt }}\left(D^{k}(f)\right)$ then for any $\sigma \in S_{k-1}$,

$$
\sigma_{\#}\left(\pi_{\#}^{k}(c)\right)=\pi_{\#}^{k} i(\sigma)_{\#}(c)=\pi_{\#}^{k}(\operatorname{sign}(i(\sigma)) c)=\operatorname{sign}(\sigma) \pi_{\#}^{k}(c) .
$$

Thus $\pi_{\#}^{k}(c) \in C_{\ell}^{\text {Alt }}\left(D^{k-1}(f)\right)$.
Proposition 5.9. $\pi_{\#}^{k-1} \circ \pi_{\#}^{k}=0$ on $C_{\bullet}^{A l t}\left(D^{k}(f)\right)$, and $f_{\#} \pi_{\#}^{2}=0$ on $C_{\bullet}^{\text {Alt }}\left(D^{2}(f)\right)$.
Proof. Let $\sigma \in S_{k}$ be the transposition $(k-1 k)$. Clearly $\pi^{k-1} \circ \pi^{k}=\pi^{k-1} \circ \pi_{k} \circ \sigma$, and it follows that for $c \in C_{\ell}^{\text {Alt }}\left(D^{k}(f)\right)$,

$$
\left(\pi^{k-1} \circ \pi^{k}\right)_{\#}(c)=\left(\pi^{k-1} \circ \pi_{k}\right)_{\#}\left(\sigma_{\#}(c)\right)=\left(\pi^{k-1} \circ \pi^{k}\right)_{\#}(-c)=-\left(\pi^{k-1} \circ \pi^{k}\right)_{\#}(c) .
$$

Since $C_{\ell}^{\text {Alt }}\left(D^{k-2}(f)\right.$ is free abelian, this proves that $\left(\pi^{k-1} \circ \pi^{k}\right) \#(c)=0$.
The second statement is proved by essentially the same argument.
Suppose that $c_{2} \in C_{\ell}^{\text {Alt }}\left(D^{2}(f)\right)$ represents a homology class in $H_{\ell}^{\text {Alt }}\left(D^{2}(f)\right)$. Then $\pi_{\#}^{2}\left(c_{2}\right)$ is also closed in $C \cdot(X)$. Now let us make the assumption that $H_{\ell}(X)=0$. This is certainly justified if $X$ is the (contractible) domain of a stable perturbation of a corank 1 map-germ. The assumption also tallies with the evidence provided by Propositions 5.6 and 5.7 in the case of a stable perturbation of a corank 1 map-germ, for these suggest (though they do not prove) that if $H_{\ell}^{\text {Alt }}\left(D^{k}\left(f_{t}\right)\right) \neq 0$ then $H_{\ell}^{\text {Alt }}\left(D^{k-1}\left(f_{t}\right)\right)=0$. We make it now in order to motivate a later more formal construction.

We will refer to this assumption as the Vanishing Assumption.
Under this assumption, since $\pi_{\#}^{2}\left(c_{2}\right)$ is a cycle, it must also be a boundary: there exists $c_{1} \in$ $C_{\ell+1}(X)$ such that $\partial c_{1}=\pi_{\#}^{2}\left(c_{2}\right)$. Then $f_{\#}\left(c_{1}\right)$ is a cycle in the image of $f$, for $\partial f_{\#}\left(c_{1}\right)=f_{\#}\left(\partial c_{1}\right)=$ $f_{\#} \pi_{\#}^{2}\left(c_{2}\right)$, and this is equal to 0 by 5.9.

Conclusion: From the alternating homology class $\left[c_{2}\right] \in H_{\ell}^{\text {Alt }}\left(D^{2}(f)\right.$, under the assumption that $H_{\ell}(X)=0$, we have constructed a homology class $\left[f_{\#}\left(c_{1}\right)\right] \in H_{\ell+1}(Y)$.

Warning: We have not constructed a map $H_{\ell}^{\text {Alt }}\left(D^{2}(f) \rightarrow H_{\ell+1}(Y)\right.$; there was an element of arbitrariness in the choice of $c_{1}$. In fact if $c_{1}^{\prime}$ is any other choice of $\ell+1$-chain on $X$ such that $\partial c_{1}^{\prime}=\pi_{\#}^{2}\left(c_{2}\right)$ then $c_{1}-c_{1}^{\prime}$ represents a homology class in $H_{\ell+1}(X)$, and thus the homology classes of $f_{\#}\left(c_{1}\right)$ and $f_{\#}\left(c_{1}^{\prime}\right)$ in $H_{\ell+1}$ differ by an element of $f_{*} H_{\ell+1}(X)$. Our construction in fact yields a $\operatorname{map} H_{\ell}^{\text {Alt }}\left(D^{2}\right) \rightarrow H_{\ell+1}(Y) / f_{*} H_{\ell+1}(X)$.

Example 5.10. In this example $X$ is contractible, so the imprecision in the choice of the cycle $f_{\#}\left(c_{2}\right)$ does not arise. Consider the stable perturbation $f_{t}: \mathbb{R}^{2} \rightarrow \mathbb{R}^{3}$, defined by $f_{t}(x, y)=$ $\left(x, y^{2}, y^{3}+x^{2}+t y\right)$, of the singularity $f=f_{0}$ of type $S_{1}$. We have

$$
D^{2}\left(f_{t}\right)=\left\{\left(x, y_{1}, y_{2}\right): y_{1}+y_{2}=0=x^{2}+y_{1}^{2}+y_{1} y_{2}+y_{2}^{2}+t\right\}
$$

The projection $\pi^{2}\left(x, y_{1}, y_{2}\right)=\left(x, y_{1}\right)$ (with inverse $\left.(x, y) \mapsto(x, y,-y)\right)$ maps this isomorphically to the conic

$$
D_{1}^{2}\left(f_{t}\right):=\left\{(x, y) \in \mathbb{C}^{2}: x^{2}+y^{2}+t=0\right\}
$$

with the involution $\sigma\left(x, y_{1}, y_{2}\right)=\left(x, y_{2}, y_{1}\right)$ now induced by $(x, y) \mapsto(x,-y)$.


Let $a$ and $b$ be 1-simplices running from $U$ to $V, a$ on the upper arc and $b$ on the lower arc of $D^{2}$, such that $\sigma \circ a=b$. Then the alternating homology $H_{1}^{\text {Alt }}\left(D^{2}\left(f_{t}\right)\right.$ is generated by $a-b$. Since here $D^{2}\left(f_{t}\right)$ is embedded in the domain $X$ of $f_{t}$, we identify $a-b \in C_{1}^{\text {Alt }}\left(D^{2}\left(f_{t}\right)\right)$ with its image in $C_{1}(X)$.Taking as $c_{2}$ a suitable triangulation of the interior of the shaded disc, we have $\partial c_{2}=a-b$. As can be seen in the picture, $f_{\#}\left(c_{2}\right)$ forms a bubble whose homology class generates $H_{2}(Y)$.

In fact this picture shows all the action of the complexified map $\mathbb{C}^{2} \rightarrow \mathbb{C}^{3}$. Here $D^{2}\left(f_{t}\right) \simeq D_{1}^{2}\left(f_{t}\right)$ is the complex Milnor fibre of an $A_{1}$ singularity, and is diffeomorphic to a cylinder. However its alternating homology is generated by the cycle shown in the real picture, and from there on the construction is the same.

Exercise 5.11. 1. Check that our map $H_{\ell}^{\mathrm{Alt}}\left(D^{2}(f)\right) \rightarrow H_{\ell+1}(Y) / f_{*} H_{\ell+1}(X)$ is well-defined in the sense that if $c_{2}$ and $c_{2}^{\prime}$ represent the same alternating homology class in $H_{\ell}^{\text {Alt }}\left(D^{2}(f)\right)$ then the resulting homology classes are the same in $H_{\ell+1}(Y) / f_{*} H_{\ell+1}(X)$.
2. Show that if we dispense with the Vanishing Assumption (that $H_{\ell}(X)=0$ ), our construction yields a map

$$
\operatorname{ker}\left(\pi_{*}^{2}: H_{2}^{\mathrm{Alt}}\left(D^{2}(f)\right) \rightarrow H_{2}(X)\right) \rightarrow H_{\ell+1}(Y)
$$

3. Under the Vanishing Assumption (to simplify notation) let $F_{\ell}$ be the image of $H_{\ell}^{\text {Alt }}\left(D^{2}\right)$ in $H_{\ell+1}(Y) / f_{*} H_{\ell=1}(X)$, and let $\bar{F}_{\ell}$ be the preimage of $F_{\ell}$ in $H_{\ell+1}(Y)$. Show that if we assume also that $H_{\ell-1}^{\text {Alt }}\left(D^{2}(f)\right)=0$, the construction of the last two pages can be extended to give a map $H_{\ell-1}^{\text {Alt }}\left(D^{3}(f)\right) \rightarrow H_{\ell+1}(Y) / \bar{F}_{\ell}$. The scheme of the argument is shown in the following diagram, in which we begin with an alternating $(\ell-1)$-cycle $a_{3}$ on $D^{3}(f)$ and successively
choose $a_{2} \in C_{\ell}^{\mathrm{Alt}} D^{2}(f)$ and $a_{1} \in C_{\ell+1}(X)$.

$$
\begin{equation*}
\ell-2 \quad \ell-1 \quad \ell+1 \tag{5.14}
\end{equation*}
$$


4. Show how to modify the construction if the Vanishing Assumptions are dropped.

### 5.4 The image computing spectral sequence

The rather complicated combinatorics of the previous constructions are all bundled up together in a spectral sequence which was first described in [16] and later developed and extended in [17], [18] and [19]. The main theorems of [17] on this topic are the following. We give the first in approximate form in order not to hide its statement in a technical fog.

Theorem 5.12. Let $f: X \rightarrow Y$ be a finite surjective map of topological spaces. Then there is a spectral sequence with $E_{p q}^{1}=H_{p}^{\text {Alt }}\left(D^{q}(f)\right)$, converging to $H_{p+q-1}(Y)$.

This means that all of the homology of the image comes either from the homology of $X$, or from the alternating homology of the multiple point spaces.

Exercise 5.13. 1. Viewing $\mathbb{R}^{2}$ as the image of the upper unit disc under the map which identifies opposite points on the boundary, find an alternating homology class in $H_{0}^{\text {Alt }}\left(D^{2}(f)\right)$ which gives rise to a generator of $H_{1}\left(\mathbb{R} \mathbb{P}^{2}\right) \simeq \mathbb{Z} / 2 \mathbb{Z}$. Generalise this to $\mathbb{R} \mathbb{P}^{n}$, taking care to distinguish between the case $n$ even and $n$ odd.
2. Let $X$ be the disjoint union of 3 real lines and $f: X \rightarrow \mathbb{R}^{2}$ be the map

$$
\left\{\begin{array}{rll}
u & \mapsto & (u, 0) \\
v & \mapsto & (0, v) \\
w & \mapsto(w, 1-w)
\end{array}\right.
$$


(a) Where does the 1-cycle in the image of $f$ come from?
(b) Does complexifying $f$ into a map from the disjoint union of three complex lines into $\mathbb{C}^{2}$ make any difference?
3. Generalising the previous exercise, consider the map from the disjoint union of $n+2$ copies of $\mathbb{R}^{n}$ to $\mathbb{R}^{n+1}$, mapping the $j$ 'th copy of $\mathbb{R}^{n}$ to the coordinate plane $\left\{x_{j}=0\right\}$ for $j=1, \ldots, n+1$ and mapping the last copy of $\mathbb{R}^{n}$ by

$$
\left(x_{1}, \ldots, x_{n}\right) \mapsto\left(x_{1}, \ldots, x_{n},\left(1-\sum_{i} x_{i}\right)\right) .
$$

The image, $Y$, is the boundary of an $n+1$ simplex, and topologically a sphere. Where does the $n$ - cycle generating $H_{n}(Y)$ come from?

Corollary 5.14. Suppose that $f_{t}: X_{t} \longrightarrow Y_{t}$ is a stable perturbation of a corank 1 map germ $f:\left(\mathbb{C}^{n}, 0\right) \rightarrow\left(\mathbb{C}^{n+r}, 0\right)$.

1. If $r \geq 2$, then

$$
H_{q}\left(Y_{t}\right)= \begin{cases}H_{n-(k-1) r}^{\text {Alt }}\left(D^{k}\left(f_{t}\right)\right) & \text { if } q=n-(k-1)(r-1) \text { for some } k \\ \mathbb{Z} & \text { if } q=0 \\ 0 & \text { otherwise }\end{cases}
$$

2. If $r=1$, then $H_{q}(Y)=0$ if $q \neq 0, n$, and there is a filtration on $H_{n}\left(Y_{t}\right)$ such that the associated graded module is isomorphic to the direct sum

$$
\bigoplus_{k=2}^{n+1} H_{n-k+1}^{A l t}\left(D^{k}\left(f_{t}\right)\right)
$$

If $D^{k}$ is an $S_{k}$-invariant ICIS of dimension $r$ with $S_{k}$-invariant Milnor fibre $D_{t}^{k}$, let us refer to the rank of $H_{r}^{\text {Alt }}\left(D_{t}^{k}\right)$ as the alternating Milnor number of $D^{k}$. Then we have

Corollary 5.15. In the situation of 5.14(2), the image Milnor number of $f$ is the sum of the alternating Milnor numbers of the ICISs $D^{k}(f)$ for $k=2, \ldots, n+1$.

If $f:\left(\mathbb{C}^{n}, S\right) \rightarrow\left(\mathbb{C}^{n+r}, 0\right)$ is no longer assumed to have corank 1 , then we know very little about its multiple point spaces $D^{k}(f)$ and those of a stable perturbation $f_{t}$. In particular, $D^{k}(f)$ is not in general an ICIS, and, if the dimensions $(n, n+r)$ are such that there may be corank 2 stable singularities of maps $\mathbb{C}^{n} \rightarrow \mathbb{C}^{n+r}$, then $D^{k}\left(f_{t}\right)$ is not in general a smoothing of $D^{k}(f)$. Nevertheless, Kevin Houston showed in [18] that the conclusion of Theorem 5.14 still holds. The main step in the proof is the following.

Theorem 5.16. Let $f_{t}$ be a stable perturbation of a finitely determined map-germ $\left(\mathbb{C}^{n}, 0\right) \rightarrow\left(\mathbb{C}^{n+r}, 0\right)$. Then $H_{q}^{\text {Alt }}\left(D^{k}\left(f_{t}\right)\right)=0$ if $q \neq \operatorname{dim} D^{k}\left(f_{t}\right)$.

### 5.5 Open questions:

1. Theorem 5.16 is proved by a rather complicated argument using equivariant stratified Morse theory. This remarkable theorem has not received the attention it deserves, in part because the published version is hard to read and suffers from some unfortunate typography. It would be a worthwhile project to write a clearer account. Houston's philosophical motivation for the theorem is worth describing because it is simple and illuminating. The difficulty in describing $D^{k}(f)$ is entirely due to the need to remove the diagonals, by which $D^{k}(f)$ differs from the simple minded scheme

$$
(X / Y)^{k}:=X \times_{Y} X \times_{Y} \cdots \times_{Y} X:=\left\{\left(x_{1}, \ldots, x_{k}\right) \in X^{k}: f\left(x_{i}\right)=f\left(x_{j}\right) \text { for all } i, j\right\} .
$$

Away from these diagonals, $(X / Y)^{k}$ is a complete intersection, defined in $X^{k}$ by the $(k-1) p$ equations $f_{k}\left(x_{1}\right)=f_{k}\left(x_{i}\right)$ for $1 \leq k \leq p$ and $2 \leq i \leq k$. Indeed, if $f$ is finitely determined, then $(X / Y)^{k}$ is non-singular away from the diagonals, since at all genuine $k$-tuple points, which by the conic structure theorem occur away from 0 , the corresponding multi-germ of $f$ is stable. Now in the alternating chain complexes $C_{\bullet}^{\text {Alt }}(f)$ and $C_{\bullet}^{\text {Alt }}\left(D^{k}\left(f_{t}\right)\right)$, the support of no chain can contain any simplex $c$ lying entirely in any diagonal $\left\{x_{i}=x_{j}\right\}$, since, evidently, the transposition $(i, j)$ leaves $c$ fixed. It follows that for the alternating homology, $D^{k}(f)$ ought to behave like a complete intersection with isolated singularity, and new cycles should appear only in middle dimension. The extent to which this argument can be turned into a proof is not clear!
2. How can one compute the "alternating Milnor number" of $D^{k}(f)$ when $f$ has corank $>1$ ?
3. How can one compute the image Milnor number of a map-germ $\left(\mathbb{C}^{n}, S\right) \rightarrow\left(\mathbb{C}^{n+1}, 0\right)$ ? An answer to 1 ., together with Corollary 5.15 , would provide a method; beyond this, there is only a conjectural method which is part of the "Mond Conjecture", that

$$
\mu_{I}(f)=\operatorname{dim}_{\mathbb{C}} \frac{\theta(i)}{t i\left(\theta_{p}\right)+i^{*}(\operatorname{Der}(-\log h))}
$$

4. How can we find equations for $D^{3}(f)$, and higher multiple point spaces, when $f$ has corank greater than 1 ?

## 6 Multiple points in the target

By the Preparation Theorem, if $\left.f:\left(\mathbb{C}^{n}, S\right) \rightarrow \mathbb{C}^{n+1}, 0\right)$ is a finite map-germ then $\mathcal{O}_{\mathbb{C}^{n}, 0}$ is a finite module over $\mathcal{O}_{\mathbb{C}^{n+1}, 0}$. A presentation of $\mathcal{O}_{\mathbb{C}^{n}, S}$ as $\mathcal{O}_{\mathbb{C}^{n+1}, 0}$-module is an exact sequence

$$
\begin{equation*}
\mathcal{O}_{\mathbb{C}^{n+1}, 0}^{p} \xrightarrow{\lambda} \mathcal{O}_{\mathbb{C}^{n+1}, 0}^{q} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{C}^{n}, S} \longrightarrow 0 . \tag{6.1}
\end{equation*}
$$

From a presentation one can learn a great deal about the geometry of the map $f$. Indeed in principle one can learn everything, since from the presentation one can obtain an equation for the image, and from this equation once can, in principle, determine the $f$ itself, up to isomorphism, since it is the normalisation of its image. Other information, in the form of the Fitting Ideals, can be derived more immediately. We return to this after first developing an algorithm for finding a presentation.

Note that $\mathcal{O}_{\mathbb{C}^{n}, S}=\oplus_{x \in S} \mathcal{O}_{\mathbb{C}^{n}, x}$, and so if $\lambda_{x}$ is the matrix of a presentation of $\mathcal{O}_{\mathbb{C}^{n}, x}$ over $\mathcal{O}_{\mathbb{C}^{n+1}, 0}$, then the block diagonal matrix $\oplus_{x \in S} \lambda_{x}$ presents $\mathcal{O}_{\mathbb{C}^{n}, S}$ over $\mathcal{O}_{\mathbb{C}^{n+1}, 0}$. So it is enough to develop a procedure to find each local presentation $\lambda_{x}$. In what follows we take $x=0 \in \mathbb{C}^{n}$.

### 6.1 Procedure for finding a presentation:

Nakayama's Lemma tells us that if $g_{1}, \ldots, g_{m} \in \mathcal{O}_{\left.\mathbb{C}^{n}, 0\right)}$ project to a $\mathbb{C}$-basis for $\mathcal{O}_{\mathbb{C}^{n}, 0} / f^{*} \mathfrak{m}_{\mathbb{C}^{n+1}, 0}$, then $g_{1}, \ldots, g_{m}$ form a minimal set of generators for $\mathcal{O}_{\mathbb{C}^{n}, 0}$ over $\mathcal{O}_{\mathbb{C}^{n+}, 0}$. The structure of $\mathcal{O}_{\mathbb{C}^{n}, 0}$ as $\mathcal{O}_{\mathbb{C}^{n+1}, 0}$-module is determined by the relations between these generators. The fact that the $g_{i}$ generate $\mathcal{O}_{\mathbb{C}^{n}, 0}$ over $\mathcal{O}_{\mathbb{C}^{n+1}, 0}$ is equivalent to the surjectivity of

$$
\mathcal{O}_{\mathbb{C}^{n+1}, 0}^{m} \xrightarrow{\mathrm{~g}} \mathcal{O}_{\mathbb{C}^{n}, 0},
$$

where $\mathbf{g}$ sends the $i$-th basis vector $e_{i}$ to $g_{i}$. The module of relations between the $g_{i}$ is the kernel of $\mathbf{g}$, and because $\mathcal{O}_{\mathbb{C}^{n+1}, 0}$ is Noetherian, it is finitely generated. Thus there is an $m \times r$ matrix $\lambda$ over $\mathcal{O}_{\mathbb{C}^{n+1}, 0}$ such that

$$
\begin{equation*}
\mathcal{O}_{\mathbb{C}^{n+1}, 0}^{r} \xrightarrow{\lambda} \mathcal{O}_{\mathbb{C}^{n+1}, 0}^{m} \xrightarrow{\mathrm{~g}} \mathcal{O}_{\mathbb{C}^{n}, 0} \longrightarrow 0 \tag{6.2}
\end{equation*}
$$

is exact. Because the $g_{i}$ form a minimal generating set for $\mathcal{O}_{\mathbb{C}^{n}, 0}$, all entries in $\lambda$ lie in the maximal ideal of $\mathcal{O}_{\mathbb{C}^{n+1}, 0}$. Thus (6.2) is the beginning of a minimal free resolution of $\mathcal{O}_{\mathbb{T}^{n}, 0}$ over $\mathcal{O}_{\mathbb{C}^{n+1}, 0}$. The Auslander-Buchsbaum formula (see e.g. [34, Chapter ?] or [12, Chapter 19]) tells us that if $p$ is the length of such a free resolution (the projective dimension of $\mathcal{O}_{\mathbb{C}^{n}, 0}$ as $\mathcal{O}_{\mathbb{C}^{n+1}, 0}$-module) , then $p+\operatorname{depth}_{\mathcal{O}_{\mathbb{C}^{n+1}, 0}} \mathcal{O}_{\mathbb{C}^{n}, 0}=\operatorname{depth}_{\mathcal{O}_{\mathbb{C}^{n+1}, 0}} \mathcal{O}_{\mathbb{C}^{n+1}, 0}$; it follows that $p=1$. In other words, $\lambda$ may be chosen injective. This forces $r$ to be equal to $m$; for tensoring the exact sequence

$$
0 \longrightarrow \mathcal{O}_{\mathbb{C}^{n+1}, 0}^{r} \xrightarrow{\lambda} \mathcal{O}_{\mathbb{C}^{n+1}, 0}^{m} \xrightarrow{\mathrm{~g}} \mathcal{O}_{\mathbb{C}^{n}, 0} \longrightarrow 0
$$

with the field of fractions of $\mathcal{O}_{\mathbb{C}^{n+1}, 0}$ (the field $\mathcal{M}=\mathcal{M}_{\mathbb{C}^{n+1}, 0}$ of meromorphic functions), we retain exactness while killing $\mathcal{O}_{\mathbb{C}^{n}, 0}$, and thus get an exact sequence $0 \longrightarrow \mathcal{M}^{r} \longrightarrow \mathcal{M}^{m} \longrightarrow 0$.

To find a matrix $\lambda$, one can use the following procedure:

1. Choose a projection $\pi: \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n}$ such that $\pi \circ f$ is finite. A suitable projection always exists. In practice this usually means selecting $n$ of the $n+1$ component functions of $f$, though in principle it may be that none of these coordinate projections is finite. In what follows we will assume that coordinates are chosen so that $\pi\left(y_{1}, \ldots, y_{n+1}\right)=\left(y_{1}, \ldots, y_{n}\right)$.
2. Then $\mathcal{O}_{\mathbb{C}^{n}, 0}$ (source) is free over $\mathcal{O}_{\mathbb{C}^{n}, 0}$ (target); let $g_{0}, \ldots, g_{d}$ be a basis. Once again, by Nakayama's Lemma it is sufficient that the $g_{i}$ form a $\mathbb{C}$-vector-space basis for $\mathcal{O}_{\mathbb{C}^{n}, 0} /(\pi \circ$ $f)^{*} \mathfrak{m}_{\mathbb{C}^{n}, 0}$, which is finite dimensional by finiteness of $\pi \circ f$. One of the $g_{i}$ at least must be $a$ unit in $\mathcal{O}_{\mathbb{C}^{n}, 0}$; we take $g_{0}=1$.
3. Find $\lambda_{j}^{i} \in \mathcal{O}_{\mathbb{C}^{n+1}, 0}$ such that

$$
\begin{align*}
& f_{n+1}=\lambda_{0}^{0} g_{0}+\cdots+\lambda_{0}^{m} g_{m} \\
& g_{1} f_{n+1}=\lambda_{1}^{0} g_{0}+\cdots+\lambda_{1}^{m} g_{m} \\
& \ldots=\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots  \tag{6.3}\\
& g_{m} f_{n+1}=\lambda_{m}^{0} g_{0}+\cdots+\lambda_{m}^{m} g_{m}
\end{align*}
$$

Since $f_{n+1}=y_{n+1} \circ f,(6.3)$ can be rewritten as

$$
\left.\begin{array}{rccccccc}
0 & = & \left(\lambda_{0}^{0}-y_{n+1}\right) g_{0} & + & \cdots & + & \cdots & + \\
0 & = & \lambda_{1}^{0} g_{0} & + & \left(\lambda_{1}^{1}-y_{n+1}\right) g_{1} & + & \cdots & + \\
\cdots & = & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \tag{6.4}
\end{array}\right] \quad \lambda_{0}^{m} g_{m}
$$

Thus the columns of the matrix

$$
\left(\begin{array}{cccc}
\lambda_{0}^{0}-y_{n+1} & \lambda_{1}^{0} & \cdots & \lambda_{m}^{0}  \tag{6.5}\\
\lambda_{0}^{1} & \lambda_{1}^{1}-y_{n+1} & \cdots & \lambda_{m}^{1} \\
\cdots & \cdots & \cdots & \cdots \\
\lambda_{0}^{m} & \lambda_{1}^{m} & \cdots & \lambda_{m}^{m}-y_{n+1}
\end{array}\right)
$$

are relations between the $g_{i}$.
Proposition 6.1. (6.5) is the matrix of a presentation of $\mathcal{O}_{\mathbb{C}^{n}, 0}$ over $\mathcal{O}_{\mathbb{C}^{n+1}, 0}$. In other words, the columns of (6.5) generate all the relations among the $g_{i}$ over $\mathcal{O}_{\mathbb{C}^{n+1}, 0}$.
Proof. A useful trick is described in [40, 2.2]: embed $\mathbb{C}^{n}$ as the hyperplane $\{t=0\}$ in $\mathbb{C}^{n} \times \mathbb{C}$, and define $F: \mathbb{C}^{n} \times \mathbb{C} \rightarrow \mathbb{C}^{n+1}$ by

$$
F(x, t)=\left(f_{1}(x), \ldots, f_{n}(x), f_{n+1}(x)-t\right) .
$$

Write $S$ for $\mathbb{C}^{n} \times \mathbb{C}$ (source) and $T$ for $\mathbb{C}^{n+1}$ (target). Then

$$
\mathcal{O}_{S, 0} / F^{*} \mathfrak{m}_{T, 0}=\frac{\mathcal{O}_{S, 0}}{\left(f_{1}, \ldots, f_{n}, f_{n+1}-t\right)} \simeq \frac{\mathcal{O}_{\mathbb{C}^{n}, 0}}{\left(f_{1}, \ldots, f_{n}\right)}
$$

so that $g_{0}, \ldots, g_{m}$ form a free $\mathcal{O}_{T, 0}$-basis for $\mathcal{O}_{S, 0}$, and thus determine an $\mathcal{O}_{T}$-isomorphism $\mathcal{O}_{T, 0}^{m+1} \xrightarrow{\varphi} \mathcal{O}_{S, 0}$. In the diagram

$[t]_{G}^{G}$ denotes the matrix of the $\mathcal{O}_{T, 0}$-linear map $\mathcal{O}_{S, 0} \xrightarrow{t} \mathcal{O}_{S, 0}$ (multiplication by $t$ ), with respect to the basis $g_{0}, \ldots, g_{m}$ of $\mathcal{O}_{S, 0}$. We have

$$
t g_{i}=\left(f_{n+1}-y_{n+1}\right) g_{i}=\lambda_{i}^{0} g_{0}+\cdots+\left(\lambda_{i}^{i}-y_{n+1}\right) g_{i}+\cdots+\lambda_{i}^{m} g_{m}
$$

and thus $[t]_{G}^{G}$ is equal to the matrix (6.5). From the commutativity of (6.6) it follows that the cokernel of (6.5) is indeed isomorphic to $\mathcal{O}_{\mathbb{C}^{n}, 0}$ as claimed.

The presentation obtained above is not necessarily minimal, since in general

$$
\operatorname{dim}_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^{n}, 0}}{f^{*} \mathfrak{m}_{\mathbb{C}^{n+1}, 0}}<\operatorname{dim}_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^{n}, 0}}{(\pi \circ f)^{*} \mathfrak{m}_{\mathbb{C}^{n}, 0}}
$$

Nevertheless it is always injective, since the determinant of (6.5) is not zero - as can easily be seen, it is a monic polynomial of degree $m+1$ in $\mathbb{C}\left\{y_{1}, \ldots, y_{n}\right\}\left[y_{n+1}\right]$.

From the square matrix $\lambda$ one can extract a great deal of information about the geometry of $f$.
Definition 6.2. Let $R^{p} \xrightarrow{\lambda} R^{q} \xrightarrow{g} M \longrightarrow 0$ be a presentation of the $R$-module $M$. The $k^{\prime}$ th Fitting ideal of $M$ as $R$-module, $\operatorname{Fitt}_{k}^{R}(M)$, or $\operatorname{simply} \operatorname{Fitt}_{k}(M)$ if it is clear which ring we are talking about, is the ideal generated by the $(q-k) \times(q-k)$ minors of $\lambda$, provided $p \geq q-k$, and is defined to be 0 if $p<q-k$ and $R$ if $q-k \leq 0$.

Exercise 6.3. The Fitting ideals are independent of the choice of presentation of $M$. Prove this by showing

1. If $R^{a} \xrightarrow{\alpha} R^{q} \xrightarrow{g} M \longrightarrow 0$ and $R^{b} \xrightarrow{\beta} R^{q} \xrightarrow{g} M \longrightarrow 0$ are presentations of the same module with respect to the same set of generators, then

$$
\min _{q-k}(\alpha)=\min _{q-k}(\beta) .
$$

2. If $R^{s} \xrightarrow{\mu} R^{t} \xrightarrow{h} M \longrightarrow 0$ is another presentation of the same module $M$, then $g+$ $h: R^{q+t} \rightarrow M$ is surjective. For each basis vector $e_{i}$ in $R^{t}$ there exists $c_{i} \in R^{q}$ such that $g\left(c_{i}\right)=h\left(e_{i}\right)$, and thus $\left(c_{i},-e_{i}\right) \in \operatorname{ker}(g+h)$. Show that the kernel of $g+h$ is generated by such pairs $\left(c_{i},-e_{i}\right)$ together with pairs $(c, 0)$ with $c \in \operatorname{ker} g$, so that there is a presentation of the form

$$
\begin{equation*}
R^{p+t} \xrightarrow{\nu} R^{q+t} \xrightarrow{g+h} M \longrightarrow 0 \tag{6.7}
\end{equation*}
$$

with

$$
\nu=\left(\begin{array}{cc}
\lambda & -c \\
0 & I_{t}
\end{array}\right) .
$$

Clearly

$$
\min _{q+t-k}(\nu)=\min _{q-k}(\lambda) .
$$

By symmetry, the kernel of $g+h$ is also generated by pairs $(0, d)$ with $d \in \operatorname{ker} h$ and pairs $\left(e_{j}, d_{j}\right)$ where $e_{j}$ is the $j$ 'th basis vector of $R^{p}$ and $g\left(e_{j}\right)=-h\left(d_{j}\right)$. By 1 , the ideals of ( $q-k+t$ )-minors are the same.

The Fitting ideals tell us a great deal about the geometry of $f$. We give two versions of this, first, one from algebraic geometry:

Proposition 6.4. $V\left(\right.$ Fitt $\left._{k}^{R}(M)\right)=\left\{x \in\right.$ Spec $R: M_{p}$ needs more than $k$ generators over $\left.R\right\}$.
In analytic geometry there are always two ways of looking at the same object. Let $\mathscr{S}$ be a coherent sheaf on the analytic space $X$. Define the ideal sheaf $\mathcal{F}_{k}(\mathscr{S})$ as the sheaf associated to the presheaf

$$
U \mapsto \operatorname{Fitt}_{k}^{\Gamma\left(U, \mathcal{O}_{X}\right)} \Gamma(U, \mathscr{S})
$$

## Proposition 6.5.

$$
V\left(\mathcal{F}_{k}(\mathscr{S})\right)=\left\{x \in X: \mathscr{S}_{x} \text { needs more than } k \text { generators over } \mathcal{O}_{X, x}\right\}
$$

By coherence, we have
Proposition 6.6. Fitt $_{k}^{\mathcal{O}_{X, x}}\left(\mathscr{S}_{x}\right)=\left(\mathcal{F}_{k}(\mathscr{S})\right)_{x}$.
is
Corollary 6.7. Let $f:\left(\mathbb{C}^{n}, 0\right) \rightarrow\left(\mathbb{C}^{n+1}, 0\right)$ be finite and analytic. Then

$$
\begin{aligned}
& V\left(\text { Fitt }_{k} \mathcal{O}^{n+1}, 0\right. \\
&\left.\left(\mathcal{O}_{\mathbb{C}^{n}, 0}\right)\right)=\left\{y \in \mathbb{C}^{n+1}: \sum_{x \in f^{-1}(y)} \text { mult }_{x}(f)>k\right\} \\
&=\left\{y \in \mathbb{C}^{n+1}: y \text { has at least } k+1 \text { preimages, counting multiplicity }\right\}
\end{aligned}
$$

In particular, $\operatorname{det} \lambda$ defines the image of $f$, and the ideal of submaximal minors of $\lambda$ defines the set of double points.
Definition 6.8. The $k$ 'th target multiple point space of $f, M_{k}(f)$, is the space $V\left(\right.$ Fitt $\left._{k} \mathcal{O}^{\mathbb{C}^{n+1}, 0}\left(\mathcal{O}_{\mathbb{C}^{n}, 0}\right)\right)$ with analytic structure given by $\mathrm{Fitt}_{k}{ }^{\mathcal{U}_{\mathbb{C}^{n+1}, 0}}\left(\mathcal{O}_{\mathbb{C}^{n}, 0}\right)$.
Example 6.9. 1 . Let $f:\left(\mathbb{C}^{2}, 0\right) \rightarrow\left(\mathbb{C}^{3}, 0\right)$ be defined by

$$
f(x, y)=\left(x, y^{3}, x y+y^{5}\right)
$$

Take $\pi\left(Y_{1}, Y_{2}, Y_{3}\right)=\left(Y_{1}, Y_{2}\right)$; then $\mathcal{O}_{\mathbb{C}^{2}, 0}$ (source) is generated over $\mathcal{O}_{\mathbb{C}^{2}, 0}$ (target) by the classes of $1, y, y^{2}$. We have

$$
\begin{aligned}
& f_{3}=x y+y^{5}=r^{5}+Y_{1} y+Y_{2} y^{2} \\
& g_{1} f_{3}=x y^{2}+y^{6}=Y_{2}^{2} 1+0 y+Y_{1} y^{2} \\
& g_{2} f_{3}=x y^{3}+y^{7}=Y_{1} Y_{2} 1+Y_{2}^{2} y+0 y^{2}
\end{aligned}
$$

so as matrix of the presentation we obtain

$$
\left(\begin{array}{ccc}
-Y_{3} & Y_{2}^{2} & Y_{1} Y_{2} \\
Y_{1} & -Y_{3} & Y_{2}^{2} \\
Y_{2} & Y_{1} & -Y_{3}
\end{array}\right)
$$

2. Let $f:\left(\mathbb{C}^{2}, 0\right) \rightarrow\left(\mathbb{C}^{3}, 0\right)$ be defined by $f\left(x_{1}, x_{2}\right)=\left(x_{1}^{2}, x_{2}^{2}, x_{1} x_{2}\right)$, and as before take $\pi\left(Y_{1}, Y_{2}, Y_{3}\right)=$ $\left(Y_{1}, Y_{2}\right)$. Then $\mathcal{O}_{\mathbb{C}^{2}, 0}$ (source) is generated over $\mathcal{O}_{\mathbb{C}^{2}, 0}$ (target) by $1, x_{1}, x_{2}, x_{1} x_{2}$. We have

$$
\begin{aligned}
& f_{3}=x_{1} x_{2}=01+0 x_{1}+0 x_{2}+1 x_{1} x_{2} \\
& g_{1} f_{3}=x_{1}^{2} x_{2}=01+0 x_{1}+Y_{1} x_{2}+0 x_{1} x_{2} \\
& g_{2} f_{3}=x_{1} x_{2}^{2}=01+Y_{2} x_{1}+0 x_{2}+0 x_{1} x_{2} \\
& g_{3} f_{3}=x_{1}^{2} x_{2}^{2}=Y_{1} Y_{2} 1+0 x_{1}+0 x_{2}+0 x_{1} x_{2}
\end{aligned}
$$

giving presentation matrix

$$
\left(\begin{array}{cccc}
-Y_{3} & 0 & 0 & Y_{1} Y_{2} \\
0 & -Y_{3} & Y_{2} & 0 \\
0 & Y_{1} & -Y_{3} & 0 \\
1 & 0 & 0 & -Y_{3}
\end{array}\right)
$$

Row and column operations transform this to

$$
\left(\begin{array}{cccc}
0 & 0 & 0 & Y_{3}^{2}-Y_{1} Y_{2} \\
0 & -Y_{3} & Y_{2} & 0 \\
0 & Y_{1} & -Y_{3} & 0 \\
1 & 0 & 0 & 0
\end{array}\right)
$$

This is now the matrix of a presentation with respect to different set of generators (Exercise: which?), of which one is, according to the first column, superfluous. Deleting it gives the minimal presentation

$$
\left(\begin{array}{ccc}
0 & 0 & Y_{3}^{2}-Y_{1} Y_{2} \\
-Y_{3} & Y_{2} & 0 \\
Y_{1} & -Y_{3} & 0
\end{array}\right)
$$

The determinant here is a square: this corresponds to the fact that $f$ is a double covering of its image.

Exercise 6.10. Find a presentation for $\mathcal{O}_{\mathbb{C}^{n}, 0}$ over $\mathcal{O}_{\mathbb{C}^{n+1}, 0}$ when

1. $\left.f:(\mathbb{C}, 0) \rightarrow \mathbb{C}^{2}, 0\right)$ is defined by $f(x)=\left(x^{2}, x^{5}\right)$;
2. $f:(\mathbb{C}, 0) \rightarrow\left(\mathbb{C}^{2}, 0\right)$ is defined by $f\left(x=\left(x^{2}, x^{2 k+1}\right)\right.$;
3. $f:\left(\mathbb{C}^{2}, 0\right) \rightarrow\left(\mathbb{C}^{3}, 0\right)$ is defined by $f(x, y)=\left(x, y^{2}, y p\left(x, y^{2}\right)\right)$;
4. $f:\left(\mathbb{C}^{2}, 0\right) \rightarrow\left(\mathbb{C}^{3}, 0\right)$ is defined by $f(x, y)=\left(x, y^{3}, x y+y^{3 k-2}\right)$;
5. $f:\left(\mathbb{C}^{6}, 0\right) \rightarrow\left(\mathbb{C}^{7}, 0\right)$ is defined by $f\left(a, b, c, d, x_{1}, x_{2}\right)=\left(a, b, c, d, x_{1}^{2}+a x_{2}, x_{2}^{2}+b x_{1}, x_{1} x_{2}+\right.$ $\left.c x_{1}+d x_{2}\right)$.

Exercise 6.11. Show that if $f:\left(\mathbb{C}^{n}, 0\right) \rightarrow\left(\mathbb{C}^{n+1}, 0\right)$ is finite and generically $k$-to- 1 onto its image, and if $\lambda$ is the matrix of a presentation of $\mathcal{O}_{\mathbb{C}^{n}, 0}$ over $\mathcal{O}_{\mathbb{C}^{n+1}, 0}$, then $\operatorname{det} \lambda$ is the $k$ 'th power of a reduced equation for the image.

Proposition 6.12. ([40]) Let $f:(\mathbb{C}, 0) \rightarrow\left(\mathbb{C}^{n+1}, 0\right)$ be finite and generically 1-1, and let $\lambda$ be the $(m+1) \times(m+1)$ matrix of a presentation of $\mathcal{O}_{\mathbb{C}^{n}, 0}$ over $\mathcal{O}_{\mathbb{C}^{n+1}, 0}$, with respect to generators $g_{0}=1, g_{1}, \ldots, g_{m}$. Then the ideal Fitt $\left(\mathcal{O}_{\mathbb{C}^{n}, 0}\right)$ is generated by the $m \times m$ minors of the matrix $\lambda^{\prime}$ obtained from $\lambda$ by deleting its first row.

Proof. The sub- $\mathcal{O}_{\mathbb{C}^{n+1}, 0}$-module of $\mathcal{O}_{\mathbb{C}^{n}, 0}$ generated by 1 can be identified with $\mathcal{O}_{D, 0}$, where $D$ is the image of $f$. Now $\lambda^{\prime}$ is a presentation of the $\mathcal{O}_{\mathbb{C}^{n+1}, 0}$-module $\mathcal{O}_{\mathbb{C}^{n}, 0} / \mathcal{O}_{D, 0}$, and the ideal generated by the $m \times m$ minors of $\lambda^{\prime}$ is the 0 -th Fitting ideal of this module. A theorem of Buchsbaum and Eisenbud ([3]) asserts that provided the codimension of the support of $\mathcal{O}_{\mathbb{C}^{n}, 0} / \mathcal{O}_{D, 0}$ is at least 2 (in fact its greatest possible value), then

$$
\operatorname{Fitt}_{0}{ }^{\mathcal{O}_{\mathbb{C}^{n+1}, 0}}\left(\mathcal{O}_{\mathbb{C}^{n}, 0} / \mathcal{O}_{D, 0}\right)=\operatorname{Ann}_{\mathcal{O}_{\mathbb{C}^{n+1}, 0}}\left(\mathcal{O}_{\mathbb{C}^{n}, 0} / \mathcal{O}_{D, 0}\right)
$$

The proof is completed by showing that because $\mathcal{O}_{\mathbb{C}^{n}, 0}$ is a ring, all of the $m \times m$ minors of $\lambda$ lie in $\operatorname{Ann}_{\mathcal{O}_{\mathbb{C}^{n+1}, 0}}\left(\mathcal{O}_{\mathbb{C}^{n}, 0} / \mathcal{O}_{D, 0}\right)$. I leave the details as a guided exercise.

Exercise 6.13. Let $m_{j}^{i}$ be the $m \times m$ minor determinant of $\lambda$ obtained by omitting row $i$ and column $j$.

1. Use Cramer's rule to show that for all $i, j, k$,

$$
\begin{equation*}
m_{j}^{i} g_{k}=m_{j}^{k} g_{i} \tag{6.8}
\end{equation*}
$$

and in particular

$$
\begin{equation*}
m_{j}^{i}=m_{j}^{0} g_{i} . \tag{6.9}
\end{equation*}
$$

2. Because $g_{i} g_{j}$ lies in $\mathcal{O}_{\mathbb{C}^{n}, 0}$ and $\mathcal{O}_{\mathbb{C}^{n}, 0}$ is generated over $\mathcal{O}_{\mathbb{C}^{n+1}, 0}$ by the $g_{k}$, there exist $\Gamma_{i j}^{k} \in$ $\mathcal{O}_{\mathbb{C}^{n+1}, 0}$ such that $g_{i} g_{j}=\sum_{k} \Gamma_{i j}^{k} g_{k}$, with $\Gamma_{i j}^{k} \in \mathcal{O}_{\mathbb{C}^{n+1}, 0}$. Use 1. to show that

$$
m_{j}^{i} g_{k}=\sum_{\ell} \Gamma_{i k}^{\ell} m_{j}^{\ell} .
$$

Exercise 6.14. 1. Find equations for the double-point locus of the image of the map-germ $f$ of type $H_{2}$, given by $f(x, y)=\left(x, y^{3}, x y+y^{5}\right)$.
2. Show that $C$ is the image of the map $t \mapsto\left(t^{4}, t^{3}, t^{5}\right)$.
3. Check that $f^{*}\left(\right.$ Fitt $\left._{1}{ }^{\mathbb{C}^{n+1}, 0}\left(\mathcal{O}_{\mathbb{C}^{n}, 0}\right)\right)$ is a principal ideal in $\mathcal{O}_{\mathbb{C}^{n}, 0}$.
4. Find the pre-image in $\mathbb{C}^{2}$ of $C$, and show that it has a singularity of type $A_{6}$ at 0 .
5. Show that the set of real points on this curve is just 0 .
6. Can you reconcile the conclusions of 2. and 4.?

The argument in the proof of 6.1 serves to prove another result:
Proposition 6.15. ([5], [40]) The matrix $\lambda$ of a presentation of $\mathcal{O}_{\mathbb{C}^{n}, 0}$ over $\mathcal{O}_{\mathbb{C}^{n+1}, 0}$ can be chosen symmetric.

Proof. We replace the diagram (6.6) by a second diagram in which the two isomorphisms of $\mathcal{O}_{\mathbb{C}^{n}} \times \mathbb{C}, 0$ (source) with $\mathcal{O}_{\mathbb{C}^{n+1}, 0}$ (target) are no longer assumed to be the same. Write $\mathcal{O}_{S, 0}:=$ $\mathcal{O}_{\mathbb{C}^{n}} \times \mathbb{C}, 0$ (source), and $\mathcal{O}_{T, 0}:=\mathcal{O}_{\mathbb{C}^{n+1}, 0}$ (target). Because $\mathcal{O}_{S, 0}$ is a Gorenstein ring, and is finite over $\mathcal{O}_{T, 0}$ (target), there is a perfect symmetric $\mathcal{O}_{T, 0}$-bilinear pairing $(\cdot, \cdot): \mathcal{O}_{S, 0} \times \mathcal{O}_{S, 0} \rightarrow \mathcal{O}_{T, 0}$.

This is a consequence of local duality. It is proved by Scheja and Storch in [44], by showing that $\operatorname{Hom}_{\mathcal{O}_{T, 0}}\left(\mathcal{O}_{S, 0}, \mathcal{O}_{T, 0}\right)$ is cyclic as $\mathcal{O}_{S, 0}$ module (where, for $s_{1}, s_{2} \in \mathcal{O}_{S, 0}$ and $\varphi \in \operatorname{Hom}_{\mathcal{O}_{T, 0}}\left(\mathcal{O}_{S, 0}, \mathcal{O}_{T, 0}\right)$, $s_{1} \cdot \varphi\left(s_{2}\right)=\varphi\left(s_{1} s_{2}\right)$ ), picking an $\mathcal{O}_{S, 0^{-}}$generator $\Phi$ for $\operatorname{Hom}_{\mathcal{O}_{T, 0}}\left(\mathcal{O}_{S, 0}, \mathcal{O}_{T, 0}\right)$, and setting

$$
\left(s_{1}, s_{2}\right)=\Phi\left(s_{1} s_{2}\right) .
$$

Because this gives a perfect pairing, for each basis $G:=g_{0}, \ldots, g_{m}$ for $\mathcal{O}_{S, 0}$ as $\mathcal{O}_{T, 0}$ module there is a dual basis $\check{G}:=\check{g}_{0}, \ldots, \check{g}_{m}$ with the property that $\left(\check{g}_{i}, g_{j}\right)=\delta_{i j}$. Let $\check{\varphi}$ be the $\mathcal{O}_{T, 0}$ isomorphism $\mathcal{O}_{T, 0}^{m+1} \rightarrow \mathcal{O}_{S, 0}$ determined by the basis $\check{G}$. Then the matrix $[t]_{G}^{\check{G}}$ is symmetric (Exercise), and, by the argument of the proof of 6.1 , is the matrix of a presentation of $\mathcal{O}_{\mathbb{C}^{n}, 0}$ over $\mathcal{O}_{\mathbb{C}^{n+1}, 0}$.

Corollary 6.16. Let $f:\left(\mathbb{C}^{n}, 0\right) \rightarrow\left(\mathbb{C}^{n+1}, 0\right)$ be finite and generically 1-1. Then $f^{*} \operatorname{Fitt}_{1}\left(\mathcal{O}_{\mathbb{C}^{n}, 0}\right)$ is a principal ideal.

Proof. Choose a symmetric presentation $\lambda$, with respect to generators $g_{0}=1, \ldots, g_{m}$. Then in the language of the proof of $6.12, \operatorname{Fitt}_{1}\left(\mathcal{O}_{\mathbb{C}^{n}, 0}\right)$ is generated by $\left(m_{0}^{0}, \ldots, m_{m}^{0}\right)$, and so $f * \operatorname{Fitt}_{1}\left(\mathcal{O}_{\mathbb{C}^{n}, 0}\right)$ is generated by $f *\left(m_{0}^{0}\right), \ldots, f^{*}\left(m_{m}^{0}\right)$. It follows by (6.9) and the symmetry of $\lambda$ that $f^{*} \operatorname{Fitt}_{1}\left(\mathcal{O}_{\mathbb{C}^{n}, 0}\right)$ is generated by $f *\left(m_{0}^{0}\right)$.

Because $\operatorname{Fitt}_{1}\left(\mathcal{O}_{\mathbb{C}^{n}, 0}\right)=\operatorname{Ann}_{\mathcal{O}_{\mathbb{C}^{n+1}, 0}}\left(\mathcal{O}_{\mathbb{C}^{n}, 0} / \mathcal{O}_{D, 0}\right)$, the ideal $\operatorname{Fitt}_{1}\left(\mathcal{O}_{\mathbb{C}^{n}, 0}\right) \mathcal{O}_{D, 0}$ is known as the conductor ideal of the ring homomorphism $\mathcal{O}_{D, 0} \rightarrow \mathcal{O}_{\mathbb{C}^{n}, 0}$. We denote it by $\mathscr{C}$. In fact $\mathscr{C}$ is also an ideal in $\mathcal{O}_{\mathbb{C}^{n}, 0}$; it is the largest ideal of $\mathcal{O}_{D, 0}$ which is also an ideal in $\mathcal{O}_{\mathbb{C}^{n}, 0}$. The last corollary shows that as an ideal in $\mathcal{O}_{\mathbb{C}^{n}, 0}, \mathscr{C}$ is principal. One can find a generator by picking a symmetric presentation $\lambda$, but there is an easier method, due, with a rather sophisticated proof, to Ragni Piene ([42]), and, with a simpler proof, to Bill Bruce and Ton Marar ([2]):

Theorem 6.17. ([2]) Let $f:\left(\mathbb{C}^{n}, 0\right) \rightarrow\left(\mathbb{C}^{n+1}, 0\right)$ be finite and generically 1-1. Let $h$ be a reduced equation for its image, and let

$$
r_{i}:=\frac{\partial\left(f_{1}, \ldots, \widehat{f}_{i}, \ldots, f_{n+1}\right)}{\partial\left(x_{1}, \ldots, x_{n}\right)}
$$

be the minor determinant of the matrix of the derivative df obtained by omitting row $i$. Then $\left(\partial h / \partial Y_{i}\right) \circ f$ is divisible by $r_{i}$ in $\mathcal{O}_{\mathbb{C}^{n}, 0}$, and the quotient generates the conductor ideal $\mathscr{C}$.

Exercise 6.18. Find a generator for the conductor when $f$ is the map of Exercise 5.3(a). Show that $D_{1}^{2}(f)$ is isomorphic to the product $\mathbb{C} \times D_{2}$, where $D_{2}$ is the image of the stable map of Example 3.16. This has an explanation! What is it?

### 6.2 Open questions

1. Do the Fitting ideals give a reasonable analytic structure to the multiple point spaces? And are these spaces well-behaved in the case of finitely determined map-germs $\left(\mathbb{C}^{n}, 0\right) \rightarrow\left(\mathbb{C}^{n+1}, 0\right)$ ? How do they behave under deformation? In particular, if $F$ is an unfolding of $f$ on parameter space $S$, then is $M_{k}(F)$ Cohen Macaulay (and therefore flat over $S$ )? Some partial answers are known, see [40], [23], [22], but for maps of corank greater than 1, nothing is known about the behaviour of $\mathrm{Fitt}_{k}^{\mathcal{C}^{n+1}, 0}\left(\mathcal{O}_{\mathbb{C}^{n}, 0}\right)$ under deformation when $k>3$. Recent improvements in computing power make more calculations possible, and new examples might clarify these questions.
2. One of the most famous open problems is the Lê Conjecture, that if $f:\left(\mathbb{C}^{2}, 0\right) \rightarrow\left(\mathbb{C}^{3}, 0\right)$ has corank 2 then it cannot be injective.

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[^0]:    ${ }^{1}$ This is true for any $t \neq 0$ when $k=\mathbb{C}$; when $k=\mathbb{R}$ it holds for $t>0$. Indeed in this case the inclusion of real in complex is a homotopy equivalence. It is an example of a "good real picture".

