Singularities and Characteristic Classes for Differentiable Maps I

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Eu gostaria de agradecer os organizadores por me convidar esta conferência maravilhosa !

What's about ?

This mini-course is about

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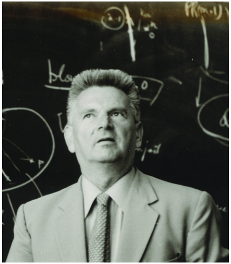


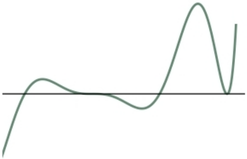
... about the polynomial named in honor of him

What's about ?

- Alg. equation over $\mathbb{C}$ $\quad(\rightsquigarrow \mathcal{K}$-classification)

$$P(x) = x^d + a_1 x^{d-1} + \cdots + a_d = 0, \qquad \sharp_{vir}\text{ sol.} = d$$

taking account of multiplicities $e = 1 + \mu$ (nondeg. sol. $\leftrightarrow \mu = 0$)



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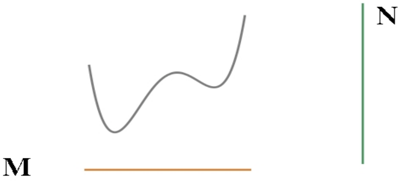
- Function $y = P(x)$ $\quad$ ($\rightsquigarrow$ $\mathcal{A}$-classification)

$$f : M \to N \qquad (M = N = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\})$$



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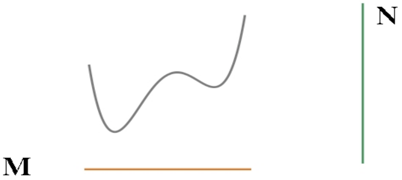

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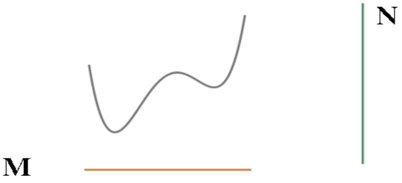

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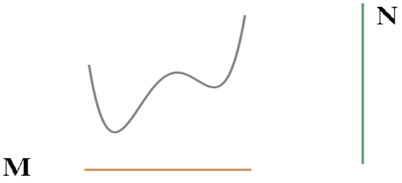

$$\sharp_{vir} \text{ crit. pt} \quad = \quad \int_M \mu(f,x)\, d\chi = 2d - 2$$

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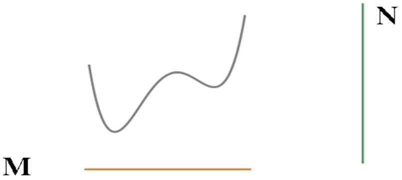

$$\sharp_{vir}\ \text{crit. pt} \quad = \quad \int_M \mu(f, x)\, d\chi = 2d - 2 = \deg f \cdot \chi(N) - \chi(M)$$

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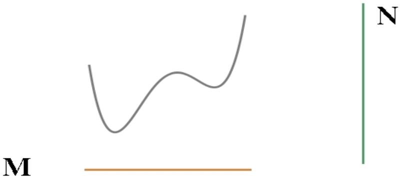

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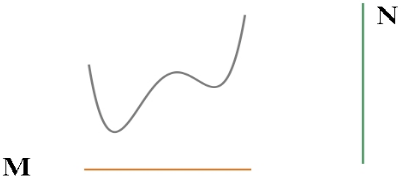

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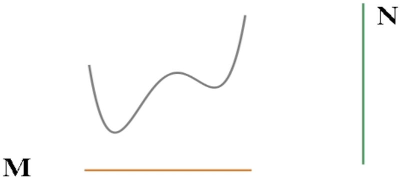

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What's about ?

I will talk about a generalization of this picture, in particular,

hunting invariants of map-germs by localizing 'higher Tp'

Contents

- Preliminary: very basics
- Thom polynomials for singularities of maps
- Thom polynomials for multi-singularities of maps
- Higher Thom polynomials associated to CSM class
- Computing numerical invariants: Bezout type theorems
- Tp for real singularities and Vassiliev type invariants

We works in the complex holomorphic context throughout.
To be elementary and self-contained as much as possible.

Classification of map-germs: Equivalence

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  Classifies map-germs up to isomorphisms of source and target
  $\mathcal{A} = \mathrm{Diff}(\mathbb{C}^m, 0) \times \mathrm{Diff}(\mathbb{C}^n, 0)$ acts on $\mathcal{O}(m,n)$ by
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  $\mathcal{K} \subset \mathrm{Diff}(\mathbb{C}^m \times \mathbb{C}^n, 0)$, preserving fibers $* \times \mathbb{C}^n$ and $\mathbb{C}^m \times 0$, acts on $\mathcal{O}(m,n)$ measuring the tangency of graph $\boldsymbol{y} = f(\boldsymbol{x})$ and $\boldsymbol{y} = 0$

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- $\mathcal{A} \subset \mathcal{K}$ Thus, orbits $\mathcal{A}.f \subset \mathcal{K}.f$

Classification of map-germs: Infinitesimal stability

- $f = (x^3 + yx, y)$ and $g = (x^3, y)$ in $\mathcal{O}(2,2)$ are $\mathcal{K}$-equivalent but not $\mathcal{A}$-equivalent. $\mathcal{A}.f \neq \mathcal{K}.f$

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- (J. Mather IV) If f is a stable germ, $\mathcal{A}.f = \{\text{Stable germs}\} \cap \mathcal{K}.f$

Classification of map-germs: Jet-extension

Given a map $f : M \to N$, we may think of it as

a family of mono-germs $f : M, x \to N, f(x)$
parameterized by the source space M.

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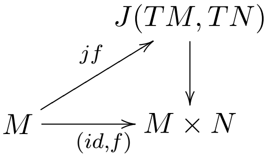

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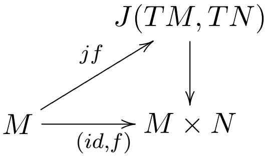
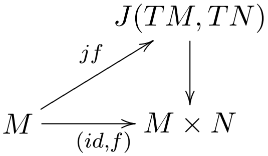


$f : M, x \to N, y$ is stable
$\iff jf : M \to J(TM, TN)$ is transverse to the $\mathcal{A}$-orbit at x.
$\iff jf : M \to J(TM, TN)$ is transverse to the $\mathcal{K}$-orbit at x (Mather)

Classification of map-germs: Jet-extension

Notation: For a $\mathcal{K}$ (or $\mathcal{A}$)-orbit η in $\mathcal{O}(m, n)$, define

$$\eta(f) := \{\ x \in M \mid \text{the germ } f \text{ at } x \text{ is of type } \eta\ \} = jf^{-1}(\eta(M, N))$$



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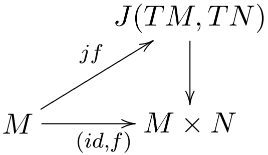

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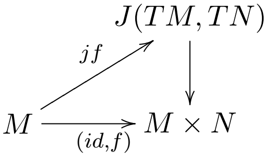

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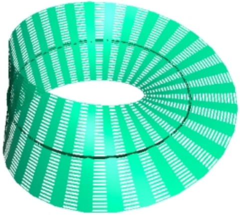
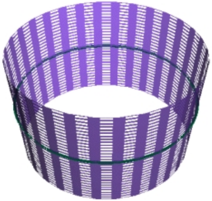
"counting η-singular points = describing this cohomology class"

Chern class of vector bundles: Definition

Recall a basic notion in topology:





A **vector bundle** $p : E \to M$ is a locally trivial fibration with fiber $\mathbb{C}^n$ and structure group GL_n.

The right one is called **the trivial bundle**.
How can we measure 'non-trivial gluing' in the left?

Chern class of vector bundles: Definition

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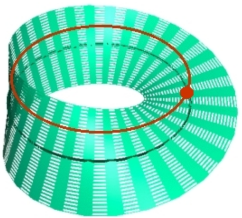

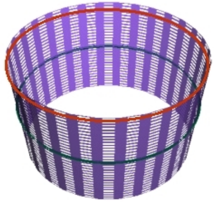

Take a section $s : M \to E$ and observe its intersection with Z, that leads us the definition of **the top Chern class of E**

$$c_n(E) := s^*\mathrm{Dual}\,[Z] = \mathrm{Dual}\,[s^{-1}(Z)] \quad \in H^{2n}(M;\mathbb{Z})$$

For the above picture, $c_n(Left) \neq 0$ and $c_n(Right) = 0$

Chern class of vector bundles: Definition

Theorem 2.1 (or Definition: see Milnor's or Hirzebruch's textbooks.)

The Chern class of complex vector bundles *is uniquely characterized as the assignment*

$$\text{vector bdle } E \to M \quad \rightsquigarrow \quad c_i(E) \in H^{2i}(M;\mathbb{Z}), \quad (i = 0, 1, 2, \cdots)$$

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satisfying the following axioms:

- $c_0(E) = 1$ *and* $c_i(E) = 0 \ \ (i > n = \operatorname{rank} E)$, *i.e.,*

  $c(E) := \sum_{i \geq 0} c_i(E) = 1 + c_1(E) + \cdots + c_n(E)$ *: total Chern class*

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  $$c(E) := \sum_{i \geq 0} c_i(E) = 1 + c_1(E) + \cdots + c_n(E) \quad : \text{ total Chern class}$$
- $c(f^*E) = f^*c(E)$ *for the pullback via* $f : M' \to M$ *: naturality*
- $c(E \oplus F) = c(E) \cdot c(F)$ *: Whitney sum formula*
- $c_1(\mathcal{O}_{\mathbb{P}^1}(1))$ *equals the divisor class* $a \in H^2(\mathbb{P}^1)$ *: normalization*

Chern class of vector bundles: Remark

- *Trivial bundle*: $c_1(\epsilon^1) = 0$, hence for the trivial n-bundle, $c(\epsilon^n) = c(\oplus\epsilon^1) = 1$.

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- *Tensor product* of line bundles ℓ_1, ℓ_2 over M:

$$c_1(\ell_1 \otimes \ell_2) = c_1(\ell_1) + c_1(\ell_2) \quad \text{(additive group law)}$$

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- The Chern class of a complex manifold M means $c(TM)$ of the tangent bundle. The top Chern class is the **Euler characteristic**:

$$c_n(TM) \frown [M] = \chi(M) \cdot [pt] \quad \in H_0(M)$$

That is **the Poincaré-Hopf theorem** : for a vector field $v : M \to TM$

$$c_n(TM) = \sum Ind(v,p) \overset{\text{P.H.}}{=} \chi(M)$$

Difference Chern class: To measure the difference between two vector bundles E and F over the same base space, we define by using formal expansion $\frac{1}{1+A} = 1 - A + A^2 - A^3 + \cdots$

$$c(F-E) := \frac{1 + c_1(F) + c_2(F) + \cdots}{1 + c_1(E) + c_2(E) + \cdots}$$

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Obviously,

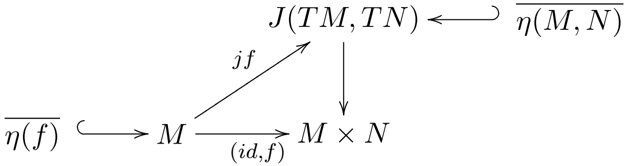
- If $F = E \oplus E'$, then $c(F - E) = c(E')$ by Whitney sum formula.

- For line bundles, $c(\ell' - \ell) = \frac{1+b}{1+a} = (1 + b)(1 - a + a^2 - \cdots)$ where $a = c_1(\ell)$ and $b = c_1(\ell')$

Thom polynomials of stable singularities

Now, return back to our setting:

Let $\eta \subset J(m,n)$ be a $\mathcal{K}$-orbit. Given a stable map $f : M \to N$,



How to describe $\quad \mathrm{Dual}\,[\overline{\eta(f)}] \in H^*(M)$

Thom polynomials of stable singularities

Theorem 3.1 (Thom ('57), Damon ('72) etc)

There exists a unique polynomial $tp(\eta) \in \mathbb{Z}[c_1, c_2, \cdots]$ *in abstract Chern classes so that*

- *homogeneous in degree* $= \operatorname{codim} \eta \quad (\deg c_i = 2i)$
- *it depends only on* $\eta \subset J(*, * + k)$,
- *for any generic map* $f : M \to N$ *of map-codim.* $\dim N - \dim M = k$, *the polynomial evaluated by* $c_i = c_i(f) := c_i(f^*TN - TM)$ *expesses the singular locus of type* η*:*

$$tp(\eta)(f) = \mathrm{Dual}\,[\overline{\eta(f)}] \quad \in H^{2\,\operatorname{codim} \eta}(M)$$

We call $tp(\eta)$ **the Thom polynomial of stable singularity type** η

Thom polynomials of stable singularities

Example 3.2 (Thom ('56): Case of map codimension $k = 0$)

Thom polynomials of stable singularities $\mathbb{C}^2, 0 \to \mathbb{C}^2, 0$ are

$$tp(A_0) = 1, \qquad tp(A_1) = c_1, \qquad tp(A_2) = c_1^2 + c_2$$

type	normal form
$A_0(regular)$	$(x, y) \mapsto (x, y)$
$A_1(fold)$	$(x, y) \mapsto (x^2, y)$
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More examples of stable singularities $\mathbb{C}^n, 0 \to \mathbb{C}^n, 0$,

$$\begin{aligned} tp(A_3) &= c_1^3 + 3c_1c_2 + 2c_3, \\ tp(A_4) &= c_1^4 + 6c_1^2c_2 + 2c_2^2 + 9c_1c_3 + 6c_4, \\ tp(I_{22}) &= c_2^2 - c_1c_3, \cdots \end{aligned}$$

Localization formula

Let's compute $tp(A_2)$ by *the restriction method* due to Richard Rimanyi. Since $\operatorname{codim} A_2 = 2$, the Thom polynomial has the form

$$tp(A_2) = Ac_1^2 + Bc_2$$

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The key point is that the normal forms of stable germs admit a natural torus action $\mathbb{C}^* = \mathbb{C} - \{0\}$:

$$(x, y) \xrightarrow{A_2} (x^3 + yx, y)$$

$$\rho_0 = \alpha \oplus \alpha^2 \qquad \rho_1 = \alpha^3 \oplus \alpha^2 \qquad \alpha \in \mathbb{C}^*$$

Let us think of α as the gluing map for the canonical line bundle $\ell = \mathcal{O}_{\mathbb{P}^N}(1)$ over $\mathbb{P}^N$ $(N \gg 0)$.

Localization formula

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That is, take $\{U_i\}$ of the base giving a local trivialization of ℓ; glueing maps $g_{ij} : U_i \cap U_j \to GL_2$ for E_0 and E_1 are of the form

$$U_i \cap U_j \xrightarrow{\alpha} \mathbb{C}^* \xrightarrow{\rho} GL_2, \quad \rho_0 = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^2 \end{bmatrix}, \quad \rho_1 = \begin{bmatrix} \alpha^3 & 0 \\ 0 & \alpha^2 \end{bmatrix},$$

respectively.

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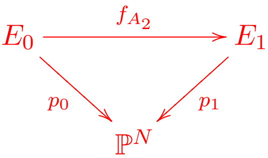
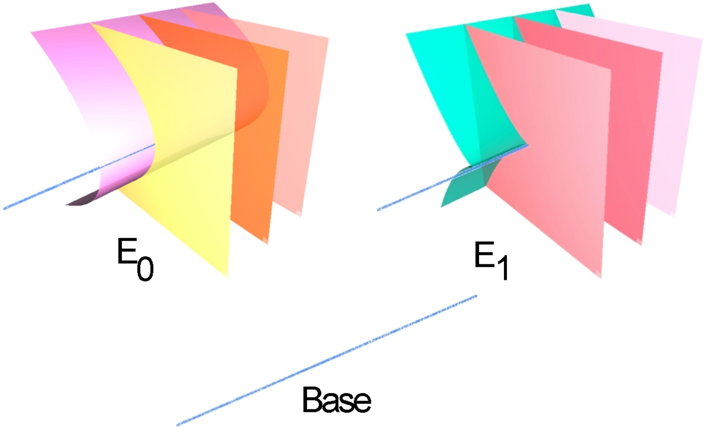
The normal form of A_2, $(x, y) \mapsto (x^3 + yx, y)$, is invariant under the action, thus we can glue the map on U_i's together. This defines a stable map $f_{A_2} : E_0 \to E_1$ between the total spaces



A_2-singularity locus $A_2(f_{A_2})$ = the zero section of E_0.

Localization formula



Localization formula

Compute the Chern classes. Put $a = c_1(\ell)$ and then

$$H^*(\mathbb{P}^N) = \mathbb{Z}[a]/(a^{N+1}), \qquad N \gg 0$$

Note that $H^*(E_0) = H^*(E_1) = H^*(\mathbb{P}^N)$ via the pullback p_0^* and p_1^*.

$$c(E_0) = c(\ell \oplus \ell^{\otimes 2}) = (1+a)(1+2a),$$
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$$\begin{aligned} c(f_{A_2}) &= c(f^*TE_1 - TE_0) = c(p_1^*E_1 - p_0^*E_0) = \tfrac{(1+3a)(1+2a)}{(1+a)(1+2a)} = \tfrac{1+3a}{1+a} \\ &= 1 + 2a - 2a^2 + 2a^3 - \cdots \end{aligned}$$

Thus we have $c_1(f_{A_2}) = 2a$, $c_2(f_{A_2}) = -2a^2$, ... etc.

Localization formula

Apply the Thom polynomial theorem to this map $f_{A_2} : E_0 \to E_1$,

$$tp(A_2)(f_{A_2}) = \mathrm{Dual}\,[\overline{A_2}(f_{A_2})]$$

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Substitute $c_2(E_0) = 2a^2, \quad c_1(f_{A_2}) = 2a, \quad c_2(f_{A_2}) = -2a^2.$

$$\begin{aligned} tp(A_2)(f_{A_2}) &= Ac_1^2 + Bc_2 \\ &= A(2a)^2 + B(-2a^2) = (4A - 2B)a^2 \\ \mathrm{Dual}\,[\overline{A_2}(f_{A_2})] &= \mathrm{Dual}\,[Zero] = c_2(E_0) = 2a^2 \end{aligned}$$

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Thus we get

$$2A - B = 1$$

Do the same thing for other singularities:

$$\begin{array}{ccc} (x,y) & \xrightarrow{A_1} & (x^2,y) \\ \circlearrowleft & & \circlearrowleft \\ \rho_0=\alpha\oplus\beta & & \rho_1=\alpha^2\oplus\beta \end{array} \qquad \alpha\in\mathbb{C}^*, \beta\in\mathbb{C}^*$$

We obtain a stable map $f_{A_1} : E_0 \to E_1$; It has only A_1-singularities, so the A_2-singularity locus $A_2(f_{A_1})$ is empty. Thus, Tp Theorem says that

$$tp(A_2)(f_{A_1}) = \mathrm{Dual}\,[\emptyset] = 0$$

Since $c(f_{A_1}) = \frac{(1+2a)(1+b)}{(1+a)(1+b)} = 1 + a - a^2 + \cdots$, one obtains

$$A - B = 0$$

Localization formula

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$$A - B = 0$$

Combine it with $2A - B = 1$, gets $A = B = 1$, i.e., $tp(A_2) = c_1^2 + c_2$

Remark 3.3

- Rimanyi's restriction method works well for simple orbits in classification up to the lowest codimension of moduli strata of orbits. In fact the restriction of tp to an orbit is the Atiyah-Bott localization for torus action (the origin is a fixed point).

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Remark 3.3

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- The universal map $f_\eta : E_0 \to E_1$ is a key ingredient in Thom-Pontrjagin-Szücs construction of classifying space of singular maps.

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- Rimanyi's restriction method works well for simple orbits in classification up to the lowest codimension of moduli strata of orbits. In fact the restriction of tp to an orbit is the Atiyah-Bott localization for torus action (the origin is a fixed point).
- The universal map $f_\eta : E_0 \to E_1$ is a key ingredient in Thom-Pontrjagin-Szücs construction of classifying space of singular maps.
- Why the difference Chern classes $c_i(f) = c_i(f^*TN - TM)$ arise ? It is that the $\mathcal{K}$-equivalence admits a stabilization of dimensions: the embedding $J(m,n) \to J(m+r, n+r)$, $jf(0) \mapsto j(f \times id_r)(0)$, is transverse to any $\mathcal{K}$-orbits (not true for $\mathcal{A}$-orbits).

Tp for $\mathcal{A}$-finite singularities

What's then about Tp for unstable but $\mathcal{A}$-finite singularities of maps?

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It makes sense.
But such a Tp is no longer a polynomial in $c_i(f)$ in general
and it's for families of maps: a proper setting should be as follows:

Consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle p_0} \searrow \qquad \swarrow {\scriptstyle p_1} & \\ & B & \end{array}$$

where X, Y, B are complex manifolds, $p_0 : X \to B$ and $p_1 : Y \to B$ are submersions of constant relative dimension, say $\dim = 2$.

Tp for $\mathcal{A}$-finite singularities

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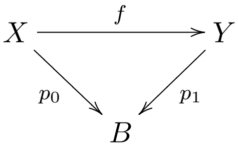

where X, Y, B are complex manifolds, $p_0 : X \to B$ and $p_1 : Y \to B$ are submersions of constant relative dimension, say $\dim = 2$.
For each $x \in X$, a map-germ of f restricted to the fiber is defined:

$$f|_{p_0^{-1}(p_0(x))} : \mathbb{C}^2, 0 \to \mathbb{C}^2, 0 \qquad \text{(centered at } x \text{ and } f(x))$$

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Given an $\mathcal{A}$-finite singularity type η, the **singularity locus** $\eta(f) \subset X$ and the **bifurcation locus** $B_\eta(f) = p_0(\eta(f)) \subset B$ are defined.

Tp for $\mathcal{A}$-finite singularities

Theorem 4.1

Let η be an $\mathcal{A}$-finite singularity type. For generic maps $f : X \to Y$, $\mathrm{Dual}\,[\overline{\eta}(f)] \in H^*(X)$ is expressed by a universal polynomial $tp^{\mathcal{A}}(\eta)$ in the Chern class $c_i = c_i(T_{X/B})$ and $c_j = c_j(T_{Y/B})$ of relative tangent bundles. $\mathrm{Dual}\,[\overline{B_\eta}(f)] \in H^*(B)$ is also expressed by the pushforward $p_{0*}tp^{\mathcal{A}}(\eta)$.

$$\begin{array}{ccccc}
\overline{\eta}(f) & \hookrightarrow & X & \xrightarrow{\ jf\ } & J(T_{X/B}, f^*T_{Y/B}) \\
\downarrow{\scriptstyle p_0} & & \downarrow{\scriptstyle p_0} & & \\
\overline{B_\eta}(f) & \hookrightarrow & B & &
\end{array}$$

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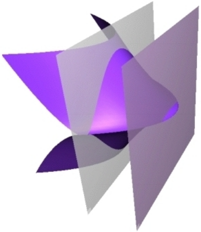
Remark 4.2

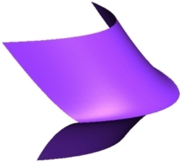
The case of rel. dim. 1: Kazarian-Lando for the study of Hurwitz numbers.

Tp for $\mathcal{A}$-finite singularities

$\mathcal{A}$-classification of $\mathbb{C}^2, 0 \to \mathbb{C}^2, 0$ (Rieger-Ruas, Arnold-Platonova)

type	codim	miniversal unfolding
lips(beaks)	3	$(x^3 + xy^2 + ax, y)$
swallowtail	3	$(x^4 + xy + ax^2, y)$
goose	4	$(x^3 + xy^3 + axy + bx, y)$
gull	4	$(x^4 + xy^2 + x^5 + axy + bx, y)$
butterfly	4	$(x^5 + xy + x^7 + ax^3 + bx^2, y)$
$I_{2,2}^{1,1}$(dertoid)	4	$(x^2 + y^3 + ay, y^2 + x^3 + bx)$





Lips

Tp for $\mathcal{A}$-finite singularities

Example 4.3 (Ohm)

Tp for $\mathcal{A}$-classification of map-germs $\mathbb{C}^2, 0 \to \mathbb{C}^2, 0$ is defined as

$$tp^{\mathcal{A}}(\eta) \in \mathbb{Z}[c_1, c_2, c_1', c_2']$$

where c_i, c_i' are Chern classes of relative tangent bundles:

lips/beaks	$-2c_1^3 + 5c_1^2c_1' - 4c_1c_1'^2 - c_1c_2 + c_2c_1' + c_1'^3$
swallowtail	$-6c_1^3 + 11c_1^2c_1' - 6c_1c_1'^2 + 7c_1c_2 - 5c_1c_2' - 5c_1'c_2 + 3c_1'c_2' + c_1'^3$
goose	$8c_1^4 - 24c_1^3c'_1 + 26c_1^2c'^2_1 - 12c_1c'^3_1 + 2c'^4_1$ $+4c_1^2c_2 - 6c_1c'_1c_2 + 2c'^2_1c_2$
gull	$6c_1^4 - 17c_1^3c'_1 + 17c_1^2c'^2_1 - 7c_1c'^3_1 + c'^4_1$ $-c_1^2c_2 + 5c_1^2c'_2 + c_1c'_1c_2 - 7c_1c'_1c'_2 + 2c'^2_1c'_2 - c_2^2 + c'^2_2$
butterfly	$24c_1^4 - 50c_1^3c'_1 - 46c_1^2c'^2_1 - 10c_1c'^3_1 + c'^4_1 - 46c_1^2c_2 + 6c_1^2c'_2$ $+60c_1c'_1c_2 - 20c_1c'_1c'_2 - 20c'^2_1c_2 + 6c'^2_1c'_2 + 3c_2^2 - 3c'^2_2$
$I_{2,2}^{1,1}$	$c_2^2 - c_1c_2c'_1 + c_2c'^2_1 + c_1^2c'_2 - 2c_2c'_2 - c_1c'_1c'_2 + c'^2_2$

Today's summary

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- Definition of Thom polynomials of stable singularities: That is a universal expression in terms of $c_i = c_i(f^*TN - TM)$ s.t.

$$tp(\eta)(f) = \mathrm{Dual}\,[\overline{\eta(f)}] \in H^*(M)$$

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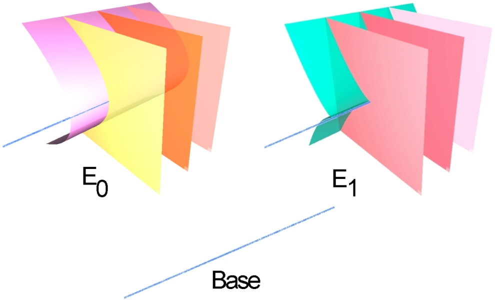
$$tp(\eta)(f) = \mathrm{Dual}\,[\overline{\eta(f)}] \in H^*(M)$$

- Torus action and computation of Tp



Até amanhã. Tchau !

ではまた明日！