Singularities and Characteristic Classes for Differentiable Maps II

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Bom dia ! おはよう！

Tudo bem ? 元気ですか？

Menu

Yesterday: main points were

- Definition of Thom polynomials for mono-singularities
- Torus action and Rimanyi's restriction method

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Today

- *Thom polynomials for multi-singularities* (M. Kazarian's theory)
- *Application:* Counting stable singularities
- *Higher Tp* based on equivariant Chern-SM class theory

Tp for multi-singularities of maps: Kazarian's theory

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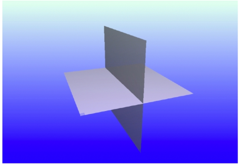
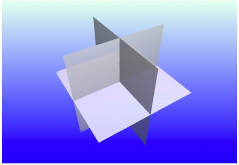
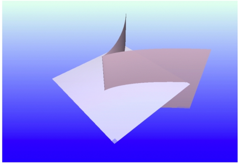
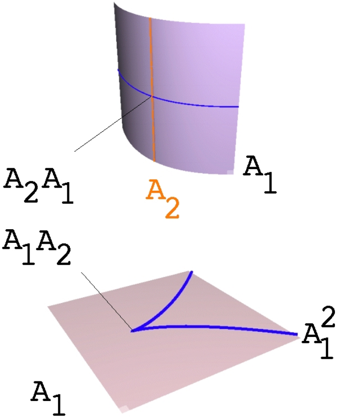
e.g., In case of $(m, n) = (3, 3)$, there are four non-mono stable types;

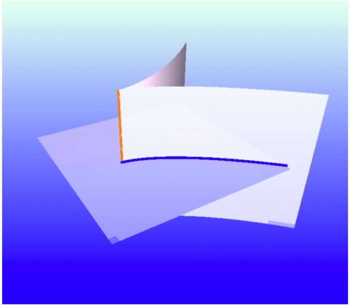
$$A_1^2 := A_1A_1, \qquad A_1^3 := A_1A_1A_1, \qquad A_1A_2, \quad A_2A_1$$

Tp for multi-singularities of maps: Kazarian's theory



Source mfd



Target mfd

Tp for multi-singularities of maps: Kazarian's theory

For $f : M \to N$, the multi-singularity loci are defined by

$$\overline{\underline{\eta}(f)} := \overline{\left\{ x_1 \in \eta_1(f) \;\middle|\; \begin{array}{c} \exists x_2, \cdots, x_r \in M \; s.t. \; x_i \neq x_j, \\ f \text{ at } x_i \text{ is of type } \eta_i \; (2 \leq i \leq r) \end{array} \right\}}$$

$$\downarrow f$$

$$\overline{f(\underline{\eta}(f))} := \overline{\left\{ y \in N \;\middle|\; \begin{array}{c} \exists x_1, \cdots, x_r \in f^{-1}(y) \; s.t. \; x_i \neq x_j, \\ f \text{ at } x_j \text{ is of type } \eta_j \; (1 \leq j \leq r) \end{array} \right\}}$$

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This is a finite-to-one map: let $\deg_1 \underline{\eta}$ be the degree

$$\deg_1 \underline{\eta} = \text{the number of } \eta_1 \text{ in the tuple } \underline{\eta}.$$

Remark that $\eta_2, \cdots, \eta_r$ could be unordered for the above def.

Tp for multi-singularities of maps: Kazarian's theory

Definition 1

The **Landweber-Novikov class** for $f: M \to N$ multi-indexed by $I = i_1 i_2 \cdots$ is

$$s_I = s_I(f) = f_*(c_1(f)^{i_1} c_2(f)^{i_2} \cdots) \quad \in H^*(N)$$

where $c_i(f) = c_i(f^*TN - TM)$.
For simplicity we often denote s_I to stand for its pullback $f^*s_I \ \in H^*(M)$ as well (i.e., omit the letter f^*).

$$s_0 = f_*(1),$$
$$s_1 = f_*(c_1),$$
$$s_2 = f_*(c_1^2), \quad s_{01} = f_*(c_2),$$
$$s_3 = f_*(c_1^3), \quad s_{11} = f_*(c_1 c_2), \quad s_{001} = f_*(c_3), \cdots$$

Tp for multi-singularities of maps: Kazarian's theory

Theorem 0.1 (M. Kazarian (2003))

Given a multi-singularity $\underline{\eta}$ *of stable-germs* $\mathbb{C}^m, 0 \to \mathbb{C}^{m+k}, 0$*, there exists a unique polynomial* $tp(\underline{\eta})$ *in abstract Chern class* c_i *and abstract Landweber-Novikov class* s_I *with rational coefficients, so that*

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- *The locus in source is expressed by the polynomial evaluated by $c_i = c_i(f) = c_i(f^*TN - TM)$ and $s_I = s_I(f) = f^*f_*(c^I(f))$:*

$$tp(\underline{\eta})(f) = \mathrm{Dual}\,[\overline{\underline{\eta}(f)}] \quad \in H^*(M;\mathbb{Q})$$

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$$tp(\underline{\eta})(f) = \mathrm{Dual}\,[\overline{\underline{\eta}(f)}] \quad \in H^*(M;\mathbb{Q})$$
- *The locus in target is expressed by a universal polynomial in* $s_I(f)$
$$tp_{\text{target}}(\underline{\eta})(f) := \frac{1}{\deg_1 \underline{\eta}}\, f_* tp(\underline{\eta}) = \mathrm{Dual}\,[\overline{f(\underline{\eta}(f))}] \quad \in H^*(N;\mathbb{Q})$$

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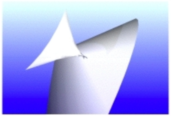
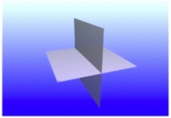
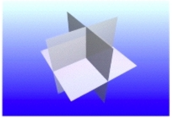
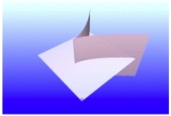
We call $tp(\underline{\eta})$ **the Thom polynomial of stable multi-singularity type** $\underline{\eta}$

Example 0.2

Tp for multi-singularities of stable maps $M^n \to N^n$ up to $\mathrm{codim}\,3$ are

type	codim	tp
A_1	1	c_1
A_2	2	$c_1^2 + c_2$
A_1A_1	2	$c_1s_1 - 4c_1^2 - 2c_2$
A_3	3	$c_1^3 + 3c_1c_2 + 2c_3$
$A_1A_1A_1$	3	$\frac{1}{2}(c_1s_1^2 - 4c_2s_1 - 4c_1s_2 - 2c_1s_{01} - 8c_1^2s_1$ $+40c_1^3 + 56c_1c_2 + 24c_3)$
A_1A_2	3	$c_1s_2 + c_1s_{01} - 6c_1^3 - 12c_1c_2 - 6c_3$
A_2A_1	3	$c_1^2s_1 + c_2s_1 - 6c_1^3 - 12c_1c_2 - 6c_3$

Tp for multi-singularities of maps: Kazarian's theory

Remark 0.3

- Kazarian's proof relies on the topology of the classifying space of complex cobordisms $\Omega^{2\infty} MU(\infty + k)$. There has not yet appeared **algebro-geometric proof** so far –

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  Göttsche conj. (thm) counting nodal curves on a surface,
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- To compute Tp for stable multi-singularities, Rimanyi's restiction method fits very well.

Application: Counting stable singularities

Here we give an application of Tp theory.

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Given a finitely determined map-germ $f : \mathbb{C}^m, 0 \to \mathbb{C}^n, 0$, and a stable (mono/multi-)singularity type η of codimension n in the target. Take a *stable perturbation*

$$f_t : U \to \mathbb{C}^n \ \ (t \in \Delta \subset \mathbb{C},\ 0 \in U \subset \mathbb{C}^m)$$

so that f_0 is a representative of f and f_t for $t \neq 0$ is a stable map. Then the number of $\eta(f_t)$ is an invariant of the original germ f.

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Ex:

$(m, 1)$: $\sharp$ Morse sing. $(A_1) \Rightarrow$ Milnor number μ.
$(2, 2)$: $\sharp$ Cusp/Double folds $\Rightarrow$ Fukuda-Ishikawa, Gaffney-Mond
$m, n \leq 8$: $\sharp$ TB singularities $\Rightarrow$ Ballesteros-Fukui-Saia ...

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USP-ICMC is the most important place about this theme !

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Let $f = (f_1, \cdots, f_n) : \mathbb{C}^m, 0 \to \mathbb{C}^n, 0$ be w. h. germs
with weights $w_1, \cdots, w_m$ and degrees $d_1, \cdots, d_n$, i.e.,

$$f(\alpha^{w_1} x_1, \cdots, \alpha^{w_m} x_m) = (\alpha^{d_1} f_1(\boldsymbol{x}), \cdots, \alpha^{d_n} f_n(\boldsymbol{x})) \quad (\forall \alpha \in \mathbb{C}^*)$$

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Suppose f is finitely determined

Given a stable mono/multi-singularity $\underline{\eta}$ of codimension n in the target.

Application: Counting stable singularities

Take a stable unfolding F of f: Suppose that unfolding parameters have weights $r_1, \cdots, r_k$.

$$\begin{array}{ccccc} & \mathbb{C}^m & \xrightarrow{f} & \mathbb{C}^n & \\ & i_0 \downarrow & & \downarrow \iota_0 & \\ \underline{\eta}(F) \subset & \mathbb{C}^{m+k} & \xrightarrow{F} & \mathbb{C}^{n+k} & \supset F(\underline{\eta}(F)) \end{array}$$

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Take a generic (non-equivariant) section ι_t close to ι_0 so that ι_t is transverse to the critical value set of F, then
it induces a stable perturbation f_t of the original map $f_0 = f$.

Take the canonical line bundle $\ell = \mathcal{O}_{\mathbb{P}^N}(1)$ over $\mathbb{P}^N$ $(N \gg 0)$ and define

$$\begin{array}{cccccc} \oplus_{i=1}^m \mathcal{O}_{\mathbb{P}^N}(w_i) =: & E_0 & \xrightarrow{f_0} & E_1 & := \oplus_{j=1}^m \mathcal{O}_{\mathbb{P}^N}(d_j) \\ & i_0 \downarrow & & \downarrow \iota_0 & \\ \underline{\eta}(F) \subset & E_0 \oplus E' & \xrightarrow{F} & E_1 \oplus E' & \supset F(\underline{\eta}(F)) \end{array}$$

where $E' = \oplus_{i=1}^k \mathcal{O}_{\mathbb{P}^N}(r_i)$ corresponding to unfolding parameters.

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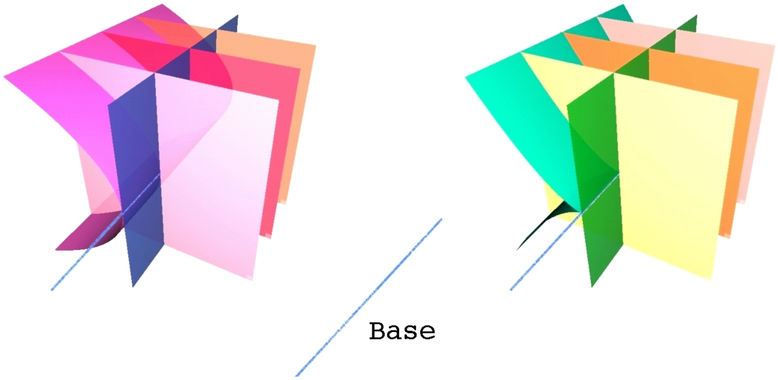
Perturb the embedding ι_0 to yield a (non-equivariant) stable perturbation $f_t : E_0 \to E_1$ of the original map $f_0 = f_\eta$.

Application: Counting stable singularities



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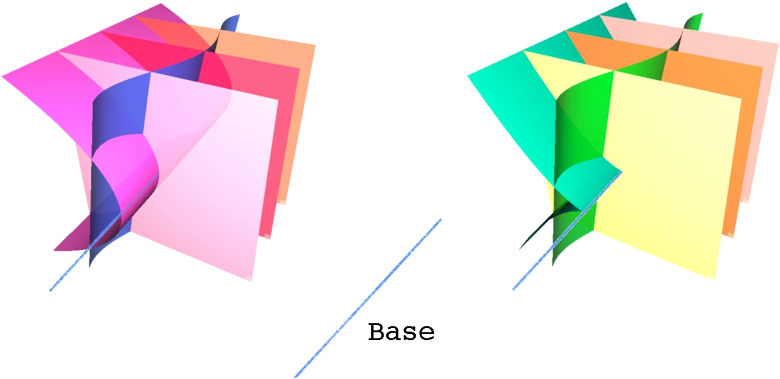

Application: Counting stable singularities

Put $a = c_1(\ell)$. In the total space $H^*(E_1 \oplus E'; \mathbb{Q}) \overset{p^*}{\simeq} H^*(\mathbb{P}^\infty; \mathbb{Q}) = \mathbb{Q}[a]$,

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$$[\overline{F(\underline{\eta}(F))}] = tp_{\text{target}}(\underline{\eta})(F) = {}^{\exists}h \cdot a^n,$$

$$[E_1] = c_{top}(p^* E') = r_1 \cdots r_k \cdot a^k.$$

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$$\sharp \underline{\eta}(f_t) = \frac{tp_{\text{target}}(\underline{\eta}) \cdot [E_1]}{c_{top}(E_1 \oplus E')} = \frac{h \cdot r_1 \cdots r_k}{d_1 \cdots d_n \cdot r_1 \cdots r_k} = \frac{h}{d_1 \cdots d_n}$$

Application: Counting stable singularities

Theorem 0.4 (Ohm)

Given a stable mono/multi-singularity $\underline{\eta}$ of codimension n in target. Then, the 0-stable invariant of a finitely determined w. h. germ $\mathbb{C}^m, 0 \to \mathbb{C}^n, 0$ is computed by

$$\sharp\underline{\eta}(f_t) = \frac{f_* tp(\underline{\eta})(f_0)}{\deg_1 \underline{\eta} \cdot d_1 \cdots d_n} = \frac{tp(\underline{\eta})(f_0)}{\deg_1 \underline{\eta} \cdot w_1 \cdots w_m}$$

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For our universal map f_0 and stable map F
$c(F) = c(f_0) = 1 + c_1(f_0) + c_2(f_0) + \cdots = \frac{\prod(1+d_j)}{\prod(1+w_i)}$ and $s_0(f_0) = \frac{d_1 \cdots d_n}{w_1 \cdots w_m}$
and $s_I(f_0) = c^I(f_0) s_0(f_0)$.
Thus the polynomial $tp(\underline{\eta})$ in $c_i = c_i(f_0)$ and $s_I = s_I(f_0)$ is written in terms of w_i and d_j.

$(m, n) = (2, 2)$: Tp of stable singularities of $\mathrm{codim}\, 2$ are

$$tp(A_2) = c_1^2 + c_2, \quad tp(A_1^2) = c_1 s_1 - 4c_1^2 - 2c_2.$$

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```
In[7]:= AC := Simplify[Expand[w1^{-1} w2^{-1} (c1^2 + c2)]]; AC
```

$$\text{Out[7]= } \left\{ \frac{d1^2 + d2^2 + 2\,w1^2 + 3\,d1\,(d2 - w1 - w2) + 3\,w1\,w2 + 2\,w2^2 - 3\,d2\,(w1 + w2)}{w1\,w2} \right\}$$

```
A := Simplify[Expand[1 / 2 d1^{-1} d2^{-1} ((d c1)^2 - 4 d c1^2  - 2 d c2 )]]; A
```

$$\left\{ \frac{1}{2\,w1^2\,w2^2} \Big(d1^3\,d2 - 2\,w1\,w2\,\big(2\,d2^2 + 3\,w1^2 + 5\,w1\,w2 + 3\,w2^2 - 5\,d2\,(w1 + w2)\big) + 2\,d1^2\,\big(d2^2 - 2\,w1\,w2 - d2\,(w1 + w2)\big) + d1\,\big(d2^3 - 2\,d2^2\,(w1 + w2) + 10\,w1\,w2\,(w1 + w2) + d2\,(w1^2 - 8\,w1\,w2 + w2^2)\big)\Big) \right\}$$

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Out[7]= { (d1^2 + d2^2 + 2 w1^2 + 3 d1 (d2 - w1 - w2) + 3 w1 w2 + 2 w2^2 - 3 d2 (w1 + w2)) / (w1 w2) }

A := Simplify[Expand[1 / 2 d1^{-1} d2^{-1} ((d c1)^2 - 4 d c1^2  - 2 d c2 )]]; A

{ 1/(2 w1^2 w2^2) (d1^3 d2 -
    2 w1 w2 (2 d2^2 + 3 w1^2 + 5 w1 w2 + 3 w2^2 - 5 d2 (w1 + w2)) + 2 d1^2 (d2^2 - 2 w1 w2 - d2 (w1 + w2)) +
    d1 (d2^3 - 2 d2^2 (w1 + w2) + 10 w1 w2 (w1 + w2) + d2 (w1^2 - 8 w1 w2 + w2^2)))}
```

This coincides with Gaffney-Mond's computation (1991).

$(m,n)=(2,3)$: Tp of stable singularities of $\operatorname{codim} 2$ in source are

$$tp(A_1)=c_2, \quad tp(A_1^3)=\frac{1}{2}(s_0^2-s_1-2c_1s_0+2c_1^2+2c_2).$$

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$$tp(A_1) = c_2, \quad tp(A_1^3) = \tfrac{1}{2}(s_0^2 - s_1 - 2c_1 s_0 + 2c_1^2 + 2c_2).$$

```
TpA1 := Simplify[Expand[w1^{-1} w2^{-1} c2]]; TPA1
```

$$\left\{\frac{-\mathrm{d3}\,\mathrm{w1} + \mathrm{w1}^2 + \mathrm{d2}\,(\mathrm{d3} - \mathrm{w1} - \mathrm{w2}) + \mathrm{d1}\,(\mathrm{d2} + \mathrm{d3} - \mathrm{w1} - \mathrm{w2}) - \mathrm{d3}\,\mathrm{w2} + \mathrm{w1}\,\mathrm{w2} + \mathrm{w2}^2}{\mathrm{w1}\,\mathrm{w2}}\right\}$$

```
TpA111 := Simplify[
  Expand[1 / 6 d1^{-1} d2^{-1} d3^{-1} (d^3 - 3 d (d c1) + 2 d (c1)^2 + 2 d c2)]]; TpA111
```

$$\Big\{\frac{1}{6\,\mathrm{w1}^3\,\mathrm{w2}^3}\Big(\mathrm{d1}^2\,\big(\mathrm{d2}^2\,\mathrm{d3}^2 - 3\,\mathrm{d2}\,\mathrm{d3}\,\mathrm{w1}\,\mathrm{w2} + 2\,\mathrm{w1}^2\,\mathrm{w2}^2\big) +$$
$$2\,\mathrm{w1}^2\,\mathrm{w2}^2\,\big(\mathrm{d2}^2 + \mathrm{d3}^2 + 2\,\mathrm{w1}^2 + 3\,\mathrm{d2}\,(\mathrm{d3} - \mathrm{w1} - \mathrm{w2}) + 3\,\mathrm{w1}\,\mathrm{w2} + 2\,\mathrm{w2}^2 - 3\,\mathrm{d3}\,(\mathrm{w1} + \mathrm{w2})\big) -$$
$$3\,\mathrm{d1}\,\mathrm{w1}\,\mathrm{w2}\,\big(\mathrm{d2}^2\,\mathrm{d3} + 2\,\mathrm{w1}\,\mathrm{w2}\,(-\mathrm{d3} + \mathrm{w1} + \mathrm{w2}) + \mathrm{d2}\,\big(\mathrm{d3}^2 - 2\,\mathrm{w1}\,\mathrm{w2} - \mathrm{d3}\,(\mathrm{w1} + \mathrm{w2})\big)\big)\Big)\Big\}$$

Application: Counting stable singularities

$(m, n) = (2, 3)$: Tp of stable singularities of $\mathrm{codim}\, 2$ in source are

$$tp(A_1) = c_2, \quad tp(A_1^3) = \tfrac{1}{2}(s_0^2 - s_1 - 2c_1 s_0 + 2c_1^2 + 2c_2).$$

```
TpA1 := Simplify[Expand[w1^{-1} w2^{-1} c2]]; TPA1
```

$$\left\{\frac{-d3\, w1 + w1^2 + d2\, (d3 - w1 - w2) + d1\, (d2 + d3 - w1 - w2) - d3\, w2 + w1\, w2 + w2^2}{w1\, w2}\right\}$$

```
TpA111 := Simplify[
  Expand[1 / 6 d1^{-1} d2^{-1} d3^{-1} (d^3 - 3 d (d c1) + 2 d (c1)^2 + 2 d c2)]]; TpA111
```

$$\Big\{\frac{1}{6\, w1^3\, w2^3} \Big(d1^2 \left(d2^2\, d3^2 - 3\, d2\, d3\, w1\, w2 + 2\, w1^2\, w2^2\right) + 2\, w1^2\, w2^2 \left(d2^2 + d3^2 + 2\, w1^2 + 3\, d2\, (d3 - w1 - w2) + 3\, w1\, w2 + 2\, w2^2 - 3\, d3\, (w1 + w2)\right) - 3\, d1\, w1\, w2 \left(d2^2\, d3 + 2\, w1\, w2\, (-d3 + w1 + w2) + d2 \left(d3^2 - 2\, w1\, w2 - d3\, (w1 + w2)\right)\right)\Big)\Big\}$$

This coincides with Mond's computation (1991).

Application: Counting stable singularities

$(m,n)=(3,3)$: Tp for $A_3 = c_1^3 + 3c_1c_2 + 2c_3$

```
In[36]:= tpA3 := c1^3 + 3 c1 c2 + 2 c3;
         tpA3 / {w1 w2 w3} /. {c1 → a1, c2 → a2, c3 → a3}
```

Out[37]= $\left\{\frac{1}{\text{w1 w2 w3}}\Big((\text{d1}+\text{d2}+\text{d3}-\text{w1}-\text{w2}-\text{w3})^3 + 3(\text{d1}+\text{d2}+\text{d3}-\text{w1}-\text{w2}-\text{w3})\big(\text{d1 d2}+(\text{d1}+\text{d2})\text{d3}-(\text{d1}+\text{d2}+\text{d3})\text{w1}+\text{w1}^2-(\text{d1}+\text{d2}+\text{d3}-\text{w1})\text{w2}+\text{w2}^2-(\text{d1}+\text{d2}+\text{d3}-\text{w1}-\text{w2})\text{w3}+\text{w3}^2\big) + 2\big(\text{d1 d2 d3}-(\text{d2 d3}+\text{d1}(\text{d2}+\text{d3}))\text{w1}+(\text{d1}+\text{d2}+\text{d3})\text{w1}^2-\text{w1}^3-\big(\text{d1 d2}+(\text{d1}+\text{d2})\text{d3}-(\text{d1}+\text{d2}+\text{d3})\text{w1}+\text{w1}^2\big)\text{w2}+(\text{d1}+\text{d2}+\text{d3}-\text{w1})\text{w2}^2-\text{w2}^3-\big(\text{d1 d2}+(\text{d1}+\text{d2})\text{d3}-(\text{d1}+\text{d2}+\text{d3})\text{w1}+\text{w1}^2-(\text{d1}+\text{d2}+\text{d3}-\text{w1})\text{w2}+\text{w2}^2\big)\text{w3}+(\text{d1}+\text{d2}+\text{d3}-\text{w1}-\text{w2})\text{w3}^2-\text{w3}^3\big)\Big)\right\}$

```
In[38]:= Factor[Simplify[% /. {d1 → w1, d2 → w2, d3 → d, w3 → w0}]]  // Simplify
```

Out[38]= $\left\{\frac{(\text{d}-3\,\text{w0})(\text{d}-2\,\text{w0})(\text{d}-\text{w0})}{\text{w0 w1 w2}}\right\}$

Our formula is valid for any corank.

In case of corank one it coincides with Marar-Montaldi-Ruas.

Application: Counting stable singularities

$(m,n) = (3,3)$: Tp for A_1^3

```
In[40]:= A1A1A1 := Simplify[
    1 / 6 (40 c1^3 + 56 c1 c2 + 24 c3 - 2 c1 s01 - 8 c1^2 s1 - 4 c2 s1 + c1 s1^2 - 4 c1 s2) (w1 w2 w3) ^ {-1} /.
     {c1 → a1, c2 → a2, c3 → a3, s0 → sa0, s1 → sa1, s01 → sa01,
      s2 → sa2, s3 → sa3, s11 → sa11, s001 → sa001}]; A1A1A1
```

Out[40]=
$$\left\{\left\{\frac{1}{6\,w1\,w2\,w3}\left(40\,(d1+d2+d3-w1-w2-w3)^3+\frac{d1^2\,d2^2\,d3^2\,(d1+d2+d3-w1-w2-w3)^3}{w1^2\,w2^2\,w3^2}-\frac{12\,d1\,d2\,d3\,(d1+d2+d3-w1-w2-w3)^3}{w1\,w2\,w3}+56\,(d1+d2+d3-w1-w2-w3)\left(d1\,d2+(d1+d2)\,d3-(d1+d2+d3)\,w1+w1^2-(d1+d2+d3-w1)\,w2+w2^2-(d1+d2+d3-w1-w2)\,w3+w3^2\right)-\frac{1}{w1\,w2\,w3}6\,d1\,d2\,d3\,(d1+d2+d3-w1-w2-w3)\left(d1\,d2+(d1+d2)\,d3-(d1+d2+d3)\,w1+w1^2-(d1+d2+d3-w1)\,w2+w2^2-(d1+d2+d3-w1-w2)\,w3+w3^2\right)+24\left(d1\,d2\,d3-(d2\,d3+d1\,(d2+d3))\,w1+(d1+d2+d3)\,w1^2-w1^3-\left(d1\,d2+(d1+d2)\,d3-(d1+d2+d3)\,w1+w1^2\right)w2+(d1+d2+d3-w1)\,w2^2-w2^3-\left(d1\,d2+(d1+d2)\,d3-(d1+d2+d3)\,w1+w1^2-(d1+d2+d3-w1)\,w2+w2^2\right)w3+(d1+d2+d3-w1-w2)\,w3^2-w3^3\right)\right)\right\}\right\}$$

```
In[41]:= Factor[Simplify[% /. {d1 → w1, d2 → w2, d3 → d, w3 → w0}]] // Simplify
```

Out[41]=
$$\left\{\left\{\frac{(d-5\,w0)\,(d-4\,w0)\,(d-3\,w0)\,(d-2\,w0)\,(d-w0)}{6\,w0^3\,w1\,w2}\right\}\right\}$$

Application: Counting stable singularities

$(m, n) = (3, 3)$: Tp for $A_1 A_2$

```
In[42]:= A1A2 := Simplify[
           (s2 c1 + s01 c1 - 6 c1^3 - 12 c1 c2 - 6 c3) (w1 w2 w3)^{-1} /.
            {c1 → a1, c2 → a2, c3 → a3, s0 → sa0, s1 → sa1, s01 → sa01,
             s2 → sa2, s3 → sa3, s11 → sa11, s001 → sa001}]; A1A2
```

Out[42]=

$$\left\{\left\{\frac{1}{w1\,w2\,w3}\left(-6\,(d1+d2+d3-w1-w2-w3)^3+\frac{d1\,d2\,d3\,(d1+d2+d3-w1-w2-w3)^3}{w1\,w2\,w3}-12\,(d1+d2+d3-w1-w2-w3)\left(d1\,d2+(d1+d2)\,d3-(d1+d2+d3)\,w1+w1^2-(d1+d2+d3-w1)\,w2+w2^2-(d1+d2+d3-w1-w2)\,w3+w3^2\right)+\frac{1}{w1\,w2\,w3}\,d1\,d2\,d3\,(d1+d2+d3-w1-w2-w3)\left(d1\,d2+(d1+d2)\,d3-(d1+d2+d3)\,w1+w1^2-(d1+d2+d3-w1)\,w2+w2^2-(d1+d2+d3-w1-w2)\,w3+w3^2\right)-6\left(d1\,d2\,d3-(d2\,d3+d1\,(d2+d3))\,w1+(d1+d2+d3)\,w1^2-w1^3-\left(d1\,d2+(d1+d2)\,d3-(d1+d2+d3)\,w1+w1^2\right)w2+(d1+d2+d3-w1)\,w2^2-w2^3-\left(d1\,d2+(d1+d2)\,d3-(d1+d2+d3)\,w1+w1^2-(d1+d2+d3-w1)\,w2+w2^2\right)w3+(d1+d2+d3-w1-w2)\,w3^2-w3^3\right)\right)\right\}\right\}$$

```
In[43]:= Factor[Simplify[Simplify[% /. {d1 → w1, d2 → w2, d3 → d, w3 → w0}]]] // Simplify
```

Out[43]=

$$\left\{\left\{\frac{(d-4\,w0)\,(d-3\,w0)\,(d-2\,w0)\,(d-w0)}{w0^2\,w1\,w2}\right\}\right\}$$

Tp for Morin maps: Porteous, Levine, Ando

Let us switch to the next theme: "Higher Tp"

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Motivation:

For Morin maps $f : M \to N$, i.e. stable maps having only A_k-singularities, the closure $S_k := \overline{A_k(f)} \stackrel{\iota}{\hookrightarrow} M$ is a closed submanifold, then

$$\iota_* c(TS_k) = \iota_*(1 + c_1(TS_k) + \cdots) \in H^*(M)$$

is sometimes thought as "higher Thom polynomials"
(Y. Ando, H. Levine, I. Porteous).
Its leading term is nothing but $\iota_*(1) = \mathrm{Dual}\,[S_k] = tp(A_k)$.

Problem

- In general (for non-Morin maps) the closure S_k is not smooth, so $c(TS_k)$ does not make sense.

Tp for Morin maps: Porteous, Levine, Ando

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- The pushforward $\iota_* \, c(TS_k) \in H^*(M)$ is not a polynomial in the difference Chern class $c_i(f) = c_i(f^*TN - TM)$
  (Here f is a Morin map).

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Solution

- We employ **Chern class for singular varieties** to develop a theory.

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(Here f is a Morin map).

Solution

- We employ **Chern class for singular varieties** to develop a theory.
- A genuine generalization of Tp must be for the **Segre class**, the Chern class of normal bundle $c(\nu)$ $(\nu = TM - TS_k)$ if S_k is smooth, not of tangent bundle.

Chern class for singular varieties

Let X be a (singular) complex algebraic variety.

Chern class for singular varieties

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$\mathcal{F}(X)$: the abelian group of <u>constructible functions</u> over X,

$$\alpha : X \to \mathbb{Z}, \quad \alpha = \sum_{\text{finite}} a_i 1\!\!1_{W_i} \quad (a_i \in \mathbb{Z} \text{ and } W_i\text{: subvarieties})$$

For proper morphisms $f : X \to Y$, we define $f_* : \mathcal{F}(X) \to \mathcal{F}(Y)$:

$$f_* 1\!\!1_W(y) := \int_{f^{-1}(y)} 1\!\!1_W := \chi(W \cap f^{-1}(y)) \qquad (y \in Y)$$

For proper $f : X \to Y$, $g : Y \to Z$, it holds that $(g \circ f)_* = g_* \circ f_*$.

$\mathcal{F} : \mathcal{V}ar \to \mathcal{A}b$: covariant functor

Chern class for singular varieties

$M^+(X)$: the group completion of the monoid generated by the isomorphism classes ($\mathcal{R}$-equiv.) of morphisms $[\rho : M \to X]$ of manifolds M to X.

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Define $\quad e[\rho : M \to X] := \rho_* 1\!\!1_M$, $\mathfrak{c}_*[\rho : M \to X] := \rho_*(c(TM) \frown [M])$

$$\begin{array}{ccc} & M^+(X) & \\ {}^{e}\swarrow & & \searrow^{\mathfrak{c}_*} \\ \mathcal{F}(X) & & H_*(X). \end{array}$$

Chern class for singular varieties

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$$\begin{array}{ccc} & M^+(X) & \\ {}^{e}\swarrow & & \searrow^{\mathfrak{c}_*} \\ \mathcal{F}(X) & \xrightarrow{\quad C_* \quad} & H_*(X). \end{array}$$

Theorem 0.5 (MacPherson (1974))

$C_* := \mathfrak{c}_* \circ e^{-1} : \mathcal{F}(X) \to H_*(X)$ *is well-defined.*

Definition 2

$C_*(X) := C_*(1\!\!1_X)$ is called the Chern-Schwartz-MacPherson class (CSM class) of X.

- M. Schwartz (1965): relative Chern class for radial vector frames
- R. MacPherson (1974): local Euler obst. + Chern-Mather
-J-P. Brasselet (1981): $c^{Sch}(X) = C_*(1\!\!1_X)$.

Remark 0.6

- Naturality: $f_* C_*(\alpha) = C_*(f_*(\alpha))$ for proper $f : X \to Y$

Chern class for singular varieties

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Chern class for singular varieties

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- Degree: For compact X, the pushforward of $pt : X \to pt$ is

$$pt_*C_*(1\!\!1_W) = \chi(W) = \int_X 1\!\!1_W.$$

In particular, $C_*(X) = \chi(X)[pt] + \cdots + [X] \in H_*(X)$

Chern class for singular varieties

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$$pt_*C_*(1\!\!1_W) = \chi(W) = \int_X 1\!\!1_W.$$
In particular, $C_*(X) = \chi(X)[pt] + \cdots + [X] \in H_*(X)$
- Exclusion-Inclusion property = Additivity:
$$C_*(1\!\!1_{A\cup B}) = C_*(1\!\!1_A) + C_*(1\!\!1_B) - C_*(1\!\!1_{A\cap B})$$

Segre-SM class

Apply C_* to a manifold: $C_* : \mathcal{F}(M) \to H^*(M)$ (omit Dual)

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For a closed submanifold $W \overset{\iota}{\hookrightarrow} M$, let ν be the normal bundle,

$$C_*(1\!\!1_W) = \iota_*(c(TW)) = \iota_*(c(\iota^*TM - \nu)) = c(TM) \cdot \iota_* c(-\nu)$$

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- inverse normal Chern classes behaves well for transverse sections:

If f is transverse to W, the fiber square gives $f^* \iota_* c(-\nu_W) = \iota'_* c(-\nu'_{W'})$

$$\begin{array}{ccc} W' & \longrightarrow & W \\ {\scriptstyle \iota'} \downarrow & & \downarrow {\scriptstyle \iota} \\ M' & \xrightarrow{f} & M \end{array}$$

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- If W is singular, $c(-\nu)$ is not defined.

Segre-SM class

Then we define

Definition 3

The **Segre-SM class** for the embedding $\iota : W \hookrightarrow M$ is defined to be

$$C_*(1\!\!1_W) = c(TM) \cdot s^{SM}(W, M) \quad \in H^*(M).$$

Also for $\alpha \in \mathcal{F}(M)$, $s^{SM}(\alpha, M)$ is defined.

- If W is smooth, $s^{SM}(W, M) = \iota_* c(-\nu)$.
- The Segre-SM class behaves well for transverse sections.

Higher Thom polynomials

Given a $\mathcal{K}$-invariant constructible function $\alpha : \mathcal{O}(m, m+k) \to \mathbb{Z}$.
For generic maps $f : M \to N$ of map-codim. k, we put a constr. ft

$$\alpha(f) : M \to \mathbb{Z}, \quad x \in M \mapsto \text{the value } \alpha \text{ of the germ } f \text{ at } x.$$

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We then define **Higher Tp** by the following thm:

Theorem 0.7 (Ohm)

There exists a universal power series $tp^{SM}(\alpha) \in \mathbb{Z}[[c_1, c_2, \cdots]]$ *so that*

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There exists a universal power series $tp^{SM}(\alpha) \in \mathbb{Z}[[c_1, c_2, \cdots]]$ so that for any generic maps $f : M \to N$, the series evaluated by $c_i = c_i(f)$ expresses the Segre-SM class $s^{SM}(\alpha(f), M)$, i.e.,

$$C_*(\alpha(f)) = c(TM) \cdot tp^{SM}(\alpha)(f) \qquad \in H^*(M)$$

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$$C_*(\alpha(f)) = c(TM) \cdot tp^{SM}(\alpha)(f) \qquad \in H^*(M)$$

Put $tp^{SM}(\overline{\eta}) := tp^{SM}(1\!\!1_{\overline{\eta}})$ for $\alpha = 1\!\!1_{\overline{\eta}}$ supported on the orbit closure.

Remark 0.8

- Since $s^{SM}(W,M) = c(TM)^{-1} \cdot C_*(1\!\!1_W) = \mathrm{Dual}\,[W] + h.o.t$, the leading term of the series is nothing but the Thom polynomial:

$$tp^{SM}(\overline{\eta}) = tp(\eta) + h.o.t$$

- For generic maps $f : M \to N$, the *Euler characteristic of the singular locus of type* η is universally expressed by

$$\chi(\overline{\eta(f)}) = \int_M c(TM) \cdot tp^{SM}(\overline{\eta})(f).$$

Remark 0.9

- To compute low dimensional terms of the universal power series $tp^{SM}(\overline{\eta})$, Rimanyi's method is very effective.

Higher Thom polynomials

Remark 0.9

- To compute low dimensional terms of the universal power series $tp^{SM}(\overline{\eta})$, Rimanyi's method is very effective.
- For singular varieties several kinds of Chern classes are available: CSM class $C_*(X)$, Chern-Mather class $c^M(X) = C_*(Eu_X)$, Fulton's Chern class and Fulton-Johnson class.
  So **Higher TP** depends on your choice.

Higher Thom polynomials

The following is a very special case: we have a closed formula:

Higher Thom polynomials

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Theorem 0.10 (Ohm)

Let $\ell = m - n \geq 0$*. Let* $\mu : \mathcal{O}(m,n) \to \mathbb{Z}$ *be the Milnor number function, which assigns to any ICIS germ its Milnor number,* 0 *otherwise.*
Then, for a stable map $f : M \to N$*, the Segre-SM class* $s^{SM}(\mu(f), M) = c(TM)^{-1}C_*(\mu(f))$ *is universally expressed by*

$$tp^{SM}(\mu) \quad = \quad (-1)^{\ell+1}(1 + c_1 + c_2 + \cdots)(\bar{c}_{\ell+1} + \bar{c}_{\ell+2} + \cdots)$$

where $c_i = c_i(f) = c_i(f^*TN - TM)$*,* $\bar{c}_i = \bar{c}_i(f) = c_i(TM - f^*TN)$*.*

Corollary 0.11 (Greuel-Hamm, Guisti, Damon, Alexandrov, etc)

Let $\eta : \mathbb{C}^m, 0 \to \mathbb{C}^n, 0$ be a weighted homogeneous ICIS, and take a universal map $f_\eta : E_0 \to E_1$ associated to its weights and degrees. Then, the Milnor number of ICIS η is expressed by

$$\mu_\eta = \int_{E_0} \frac{C_*(\mu(f_\eta))}{c_{top}(E_0)} = (-1)^{m-n}\left(\frac{c_n(E_1)}{c_m(E_0)} c_{m-n}(E_0 - E_1) - 1\right)$$

Higher Thom polynomials

Corollary 0.11 (Greuel-Hamm, Guisti, Damon, Alexandrov, etc)

Let $\eta : \mathbb{C}^m, 0 \to \mathbb{C}^n, 0$ be a weighted homogeneous ICIS, and take a universal map $f_\eta : E_0 \to E_1$ associated to its weights and degrees. Then, the Milnor number of ICIS η is expressed by

$$\mu_\eta = \int_{E_0} \frac{C_*(\mu(f_\eta))}{c_{top}(E_0)} = (-1)^{m-n} \left(\frac{c_n(E_1)}{c_m(E_0)} c_{m-n}(E_0 - E_1) - 1 \right)$$

Example 0.12

In case of $n = 1$, i.e., η is a w. h. isolated hypersurface singularity:

$$\mu_\eta = (-1)^m \frac{c_m(E_0 - E_1)}{c_m(E_0)} = \frac{top.\ (-1)^m \prod(1 + (w_i - d)t)}{top.\ \prod(1 + w_i t)} = \prod_{i=1}^{m} \frac{d - w_i}{w_i}$$

Remark 0.13

T. Suwa proved that for a complete intersection variety X with isolated singularities embedded in a manifold M, the degree of **the Milnor class**

$$\mathcal{M}(X) := (-1)^{\ell+1}(C_*(X) - c(TM|_X - \nu))$$

equals the sum of the Milnor numbers: $\mathcal{M}(X) = \int_X \mu$.

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Our formula of $tp^{SM}(\mu)$ can be applied to a T-equivariant complete intersection $X \subset M$ with weighted homogeneous isolated singularities. The degree of **Equivariant Milnor class**

$$\mathcal{M}^T(X) = (-1)^{\ell+1}(C_*^T(X) - c^T(TM|_X - \nu))$$

equals the sum of the localization of the equivariant CSM class $C_*^T(\mu) = c^T(TM) \cdot tp^{SM}(\mu)$ to singular points.

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- **Stable invariants for w. h. map-germs can be computed by localizing Tp.**
- As a higher Thom polynomial, **Segre-SM class** tp^{SM} is introduced.

Até amanhã. Tchau ! また明日！