The theory of graph-like Legendrian unfoldings and its applications

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To the memory of my friend Vladimir M. Zakalyukin.

Abstract

This is mainly a survey article on the recent development of the theory of graph-like Legendrian unfoldings and its appliations. The notion of graph-like Legendrian unfoldings belongs to a special class of the notion of big Legendrian submanifolds which was deeply investigated by Zakalyukin. Some new original results and the corrected proofs of some results are given in this paper.

1 Introduction

The notion of graph-like Legendrian unfoldings was introduced in [15] belonging to a special class of big Legendrian submanifolds which was introduced and deeply investigate by Zakalyukin[34, 35]. There have been some developments on this theory past two decades[15, 16, 24, 25, 26]. Most of the results here are implicitly or explicitly known in those articles. However, we explain the detailed proofs of those results in this survey article for understanding and applying the theory. Moreover some of the results here are original, especially Theorem 4.14 explains how the theory of graph-like Legendrian unfoldings is useful for applying to many situations related to the theory of Lagrangian singularities (caustics). Moreover, it has been known that the caustics equivalent (i.e., diffeomorphic caustics) does not mean the Lagrangian equivalence. This is one of the main differences from the theory of Legendrian singularities. In the theory of Legendrian singularities, the wave fronts equivalence (i.e., diffeomorphic wave fronts) implies the Legendrian equivalence in generic.

One of the typical examples of big wave fronts (also, graph-like wave fronts) is given by the parallels of a plane curve. For a curve in Euclidean plane, its parallels consist of those curves a fixed distance r down the normals in a fixed direction. They usually have singularities for sufficiently large r. Their singularities are always Legendrian singularities.

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It is well-known that the singularities of the parallels lie on the evolute of the curve. We can draw the picture of the parallel from an ellipse and the locus of those singularities in Fig.1. Moreover, there is another interpretation of the evolute of the curve. If we consider the family of normal lines from the curve, the evolute is the envelope of this family of the normal lines. We can also draw the envelope of the family of normal lines from an ellipse in Fig.2. The picture of the corresponding big wave front is given in Fig.3. The evolute is one of the examples of caustics and the family of parallels is one of the examples of wave front propagations.

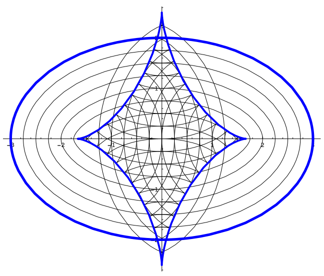


Fig.1: The parallels and the evolute of an ellipse

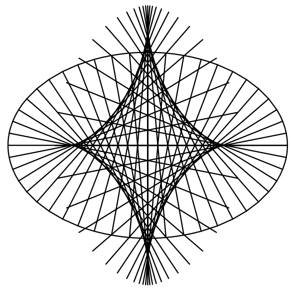


Fig.2: The normal lines and the evolute of an ellipse

Fig.3: The big wave front of the parallels of an ellipse

The caustic is described as the set of critical values of the projection a Lagrangian submanifold from the phase space onto the configuration space. In the real world, the caustics

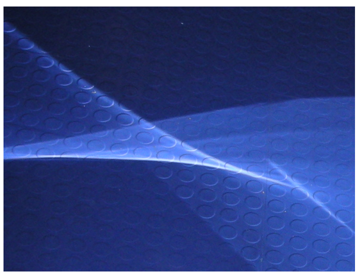


Fig.4: The caustic reflected by a mirror

given by the reflected rays are visible. However, the wave front propagations are not visible (cf. Fig. 4). Therefore, we can say that there are hidden structures (i.e., wave front propagations) on the picture of caustics. In fact, the caustic is a subject of classical physics. However, the corresponding Lagrangian submanifold is deeply related to the *semi-classical approximation* of quantum mechanics (cf., [13, 30]).

On the other hand, it was believed that the correct framework to describe the parallels of a curve is the theory of big wave fronts around 1989 [2]. But it was pointed out that A_1 and A_2 bifurcations are not appeared as the parallels of curves [3, 7]. Therefore, the framework of the theory of big wave fronts is too wide for describing the parallels of a curves. The theory of the graph-like Legendrian unfoldings was introduced to construct the correct framework for the parallels of a curves in [15]. One of the main results in the theory of graph-like Legendrian unfoldings is Theorem 4.14 which clarifies the relation between the caustics and the wave front propagations. We give some examples of applications of the theory of wave front propagations in §5.

2 Lagrangian singularities

We give a brief review on the local theory of Lagrangian singularities due to [1]. We consider the cotangent bundle $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$ over $\mathbb{R}^n$. Let $(x, p) = (x_1, \dots, x_n, p_1, \dots, p_n)$ be the canonical coordinate on $T^*\mathbb{R}^n$. Then the canonical symplectic structure on $T^*\mathbb{R}^n$ is given by the *canonical two form* $\omega = \sum_{i=1}^n dp_i \wedge dx_i$. Let $i : L \subset T^*\mathbb{R}^n$ be a submanifold. We say that i is a *Lagrangian submanifold* if $\dim L = n$ and $i^*\omega = 0$. In this case, the critical value of $\pi \circ i$ is called the *caustic* of $i : L \subset T^*\mathbb{R}^n$, which is denoted by C_L. We can interpret the evolute of a plane curve as the caustic of a certain Lagrangian submanifold (cf., §5). One of the main results in the theory of Lagrangian singularities is the description of Lagrangian submanifold germs by using families of function germs. Let $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be an n-parameter unfolding of function germs. We say that F is a *Morse family of functions* if the map germ

$$\Delta F = \left(\frac{\partial F}{\partial q_1}, \dots, \frac{\partial F}{\partial q_k} \right) : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$$

is non-singular, where $(q, x) = (q_1, \dots, q_k, x_1, \dots, x_n) \in (\mathbb{R}^k \times \mathbb{R}^n, 0)$. In this case, we have a smooth n-dimensional submanifold germ $C(F) = (\Delta F)^{-1}(0) \subset (\mathbb{R}^k \times \mathbb{R}^n, 0)$ and a map germ $L(F) : (C(F), 0) \to T^*\mathbb{R}^n$ defined by

$$L(F)(q, x) = \left(x, \frac{\partial F}{\partial x_1}(q, x), \dots, \frac{\partial F}{\partial x_n}(q, x) \right).$$

We can show that $L(F)(C(F))$ is a Lagrangian submanifold germ. Then it is known ([1], page 300) that all Lagrangian submanifold germs in $T^*\mathbb{R}^n$ are constructed by the above method. Let $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a Morse family of functions. We call F a *generating family* of $L(F)(C(F))$.

We now define a natural equivalence relation among Lagrangian submanifold germs. Let $i : (L, p) \subset (T^*\mathbb{R}^n, p)$ and $i' : (L', p') \subset (T^*\mathbb{R}^n, p')$ be Lagrangian submanifold germs. Then we say that i and i' are *Lagrangian equivalent* if there exist a diffeomorphism germ

$\sigma : (L, p) \to (L', p')$, a symplectic diffeomorphism germ $\hat{\tau} : (T^*\mathbb{R}^n, p) \to (T^*\mathbb{R}^n, p')$ and a diffeomorphism germ $\tau : (\mathbb{R}^n, \pi(p)) \to (\mathbb{R}^n, \pi(p'))$ such that $\hat{\tau} \circ i = i' \circ \sigma$ and $\pi \circ \hat{\tau} = \tau \circ \pi$, where $\pi : (T^*\mathbb{R}^n, p) \to (\mathbb{R}^n, \pi(p))$ is the canonical projection. Here $\hat{\tau}$ is a *symplectic diffeomorphism germ* if it is a diffeomorphism germ such that $\hat{\tau}^*\omega = \omega$. Then the caustic C_L is diffeomorphic to the caustic $C_{L'}$ by the diffeomorphism germ τ.

We can interpret the Lagrangian equivalence by using the notion of generating families. Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be function germs. We say that F and G are *P-$\mathcal{R}^+$-equivalent* if there exist a diffeomorphism germ $\Phi : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}^k \times \mathbb{R}^n, 0)$ of the form $\Phi(q, x) = (\phi_1(q, x), \phi_2(x))$ and a function germ $h : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ such that $G(q, x) = F(\Phi(q, x)) + h(x)$. For any $F_1 : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ and $F_2 : (\mathbb{R}^{k'} \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$, F_1 and F_2 are said to be *stably P-$\mathcal{R}^+$-equivalent* if they become P-$\mathcal{R}^+$-equivalent after the addition to the arguments q_i of new arguments q'_i and to the functions F_i of non-degenerate quadratic forms Q_i in the new arguments, i.e., $F_1 + Q_1$ and $F_2 + Q_2$ are P-$\mathcal{R}^+$-equivalent. Then we have the following theorem:

Theorem 2.1 *Let $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ and $G : (\mathbb{R}^{k'} \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be Morse families of functions. Then $L(F)(C(F))$ and $L(G)(C(G))$ are Lagrangian equivalent if and only if F and G are stably P-$\mathcal{R}^+$-equivalent.*

Let $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a Morse family of functions and $\mathcal{E}_k$ the ring of function germs of $q = (q_1, \ldots, q_k)$ variables at the origin. We say that $L(F)(C(F))$ is *Lagrangian stable* if

$$\mathcal{E}_k = J_f + \left\langle \frac{\partial F}{\partial x_1}|_{\mathbb{R}^k \times \{0\}}, \ldots, \frac{\partial F}{\partial x_n}|_{\mathbb{R}^k \times \{0\}} \right\rangle_{\mathbb{R}} + \langle 1 \rangle_{\mathbb{R}},$$

where $f = F|_{\mathbb{R}^k \times \{0\}}$ and

$$J_f = \left\langle \frac{\partial f}{\partial q_1}(q), \ldots, \frac{\partial f}{\partial q_k}(q) \right\rangle_{\mathcal{E}_k}.$$

Remark 2.2 In the theory of unfoldings[6], F is said to be an *infinitesimally P-$\mathcal{R}^+$-versal unfolding of $f = F|_{\mathbb{R}^k \times \{0\}}$* if the above condition is satisfied. There is a definition of the Lagrangian stability (cf., [1, §21.1]). It is known that $L(F)(C(F))$ is Lagrangian stable if and only if F is an infinitesimally P-$\mathcal{R}^+$-versal unfolding of $f = F|_{\mathbb{R}^k \times \{0\}}$ [1]. In this paper we do not need the original definition of the Lagrangian stability, so that we adopt the above definition.

3 Theory of the wave front propagations

In this section we give a brief survey on the theory of wave front propagations (for detail, see [1, 15, 35, 36], etc). We consider one parameter families of wave fronts and its bifurcations. The principal idea is that a one parameter family of wave fronts is considered to be a wave front whose dimension is one dimension higher than each member of the family. This is called a big wave front. Since the big wave front is a wave front, we start to consider the general theory of Legendrian singularities. Let $\overline{\pi} : PT^*(\mathbb{R}^m) \longrightarrow \mathbb{R}^m$ be the projective cotangent bundle over $\mathbb{R}^m$. This fibration can be considered as a Legendrian fibration with the canonical contact structure K on $PT^*(\mathbb{R}^m)$. We now review geometric properties of this space. Consider the tangent bundle $\tau : TPT^*(\mathbb{R}^m) \to PT^*(\mathbb{R}^m)$ and the differential map $d\overline{\pi} : TPT^*(\mathbb{R}^m) \to T\mathbb{R}^m$ of $\overline{\pi}$. For any $X \in TPT^*(\mathbb{R}^m)$, there exists an element $\alpha \in T^*(\mathbb{R}^m)$ such that $\tau(X) = [\alpha]$. For an element $V \in T_x(M)$, the property $\alpha(V) = \mathbf{0}$ does not depend on the choice of representative of the class $[\alpha]$. Thus we can define the canonical contact structure on $PT^*(\mathbb{R}^m)$ by $K = \{X \in TPT^*(\mathbb{R}^m) | \tau(X)(d\overline{\pi}(X)) = 0\}$. We have the trivialisation $PT^*(\mathbb{R}^m) \cong \mathbb{R}^m \times P(\mathbb{R}^{m*})$ and we call $(x, [\xi])$ *homogeneous coordinates*, where $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$ and $[\xi] = [\xi_1 : \cdots : \xi_m]$ are homogeneous coordinates of the dual projective space $P(\mathbb{R}^{m*})$. It is easy to show that $X \in K_{(x,[\xi])}$ if and only if $\sum_{i=1}^m \mu_i \xi_i = 0$, where $d\overline{\pi}(X) = \sum_{i=1}^n \mu_i \frac{\partial}{\partial x_i}$. Let $\Phi : (\mathbb{R}^m, 0) \longrightarrow (\mathbb{R}^m, 0)$ be a diffeomorphism germ. Then we have a unique contact diffeomorphism germ $\widehat{\Phi} : PT^*\mathbb{R}^m \longrightarrow PT^*\mathbb{R}^m$ defined by $\widehat{\Phi}(x, [\xi]) = (\Phi(x), [\xi \circ d_{\Phi(x)}(\Phi^{-1})])$. We call $\widehat{\Phi}$ the *contact lift* of Φ.

A submanifold $i : L \subset PT^*(\mathbb{R}^m)$ is said to be a *Legendrian submanfiold* if $\dim L = m-1$ and $di_p(T_pL) \subset K_{i(p)}$ for any $p \in L$. We also call the map $\overline{\pi} \circ i = \overline{\pi}|_L : L \longrightarrow \mathbb{R}^m$ the *Legendrian map* and the set $W(L) = \overline{\pi}(L)$ the *wave front* of $i : L \subset PT^*(\mathbb{R}^m)$. We say that a point $p \in L$ is a *Legendrian singular point* if $\operatorname{rank} d(\overline{\pi}|_L)_p < m - 1$. In this case $\overline{\pi}(p)$ is the singular point of $W(L)$.

The main tool of the theory of Legendrian singularities is the notion of generating families. Let $F : (\mathbb{R}^k \times \mathbb{R}^m, 0) \longrightarrow (\mathbb{R}, 0)$ be a function germ. We say that F is a *Morse family of hypersurfaces* if the mapping

$$\Delta^* F = \left(F, \frac{\partial F}{\partial q_1}, \ldots, \frac{\partial F}{\partial q_k} \right) : (\mathbb{R}^k \times \mathbb{R}^m, 0) \longrightarrow (\mathbb{R} \times \mathbb{R}^k, 0)$$

is non-singular, where $(q, x) = (q_1, \ldots, q_k, x_1, \ldots, x_m) \in (\mathbb{R}^k \times \mathbb{R}^m, 0)$. In this case we have a smooth $(m-1)$-dimensional submanifold germ

$$\Sigma_*(F) = \left\{ (q, x) \in (\mathbb{R}^k \times \mathbb{R}^n, 0) \mid F(q, x) = \frac{\partial F}{\partial q_1}(q, x) = \cdots = \frac{\partial F}{\partial q_k}(q, x) = 0 \right\}$$

and we have a map germ $\mathscr{L}_F : (\Sigma_*(F), 0) \longrightarrow PT^*\mathbb{R}^m$ defined by

$$\mathscr{L}_F(q, x) = \left(x, \left[\frac{\partial F}{\partial x_1}(q, x) : \cdots : \frac{\partial F}{\partial x_m}(q, x) \right] \right).$$

We can show that $\mathscr{L}_F(\Sigma_*(F)) \subset PT^*(\mathbb{R}^m)$ is a Lenedrian submanfiold germ. Then it is known ([1, page 320]) that all Legendrian submanifold germs in $PT^*\mathbb{R}^m$ are constructed by the above method. We call F a *generating family* of $\mathscr{L}_F(\Sigma_*(F))$. Therefore the wave front is

$$W(\mathscr{L}_F(\Sigma_*(F))) = \left\{ x \in \mathbb{R}^m \, | \exists q \in \mathbb{R}^k \text{ s.t } F(q, x) = \frac{\partial F}{\partial q_1}(q, x) = \cdots = \frac{\partial F}{\partial q_k}(q, x) = 0 \right\}.$$

Since the Legendrian submanifold germ $i : (L, p) \subset (PT^*\mathbb{R}^n, p)$ is uniquely determined on the regular part of the wave front $W(L)$, we have the following simple but significant property of Legendrian immersion germs [35].

Proposition 3.1 (Zakalyukin) *Let $i : (L, p) \subset (PT^*\mathbb{R}^m, p)$ and $i' : (L', p') \subset (PT^*\mathbb{R}^m, p')$ be Legendrian immersion germs such that regular sets of $\overline{\pi} \circ i, \overline{\pi} \circ i'$ are dense respectively. Then $(L, p) = (L', p')$ if and only if $(W(L), \overline{\pi}(p)) = (W(L'), \overline{\pi}(p'))$.*

In order to understand the ambiguity of generating families for a fixed Legendrian submanifold germ we introduce the following equivalence relation among Morse families of hypersurfaces. In the local ring $\mathcal{E}_k$ of function germs $(\mathbb{R}^k,0)\longrightarrow\mathbb{R}$, we have the unique maximal ideal $\mathfrak{M}_k=\{h\in\mathcal{E}_k \mid h(0)=0\ \}$. Let $F,G:(\mathbb{R}^k\times\mathbb{R}^m,0)\longrightarrow(\mathbb{R},)$ be Morse families of hypersurfaces. We say that F and G are *strictly parametrized $\mathcal{K}$-equivalent* (briefly, *S.P-$\mathcal{K}$-equivalent*) if there exists a diffeomorphism germ $\Psi:(\mathbb{R}^k\times\mathbb{R}^m,0)\longrightarrow(\mathbb{R}^k\times\mathbb{R}^m,0)$ of the form $\Psi(q,x)=(\psi_1(q,x),x)$ for $(q,x)\in(\mathbb{R}^k\times\mathbb{R}^m,0)$ such that $\Psi^*(\langle F\rangle_{\mathcal{E}_{k+m}})=\langle G\rangle_{\mathcal{E}_{k+m}}$. Here $\Psi^*:\mathcal{E}_{k+m}\longrightarrow\mathcal{E}_{k+m}$ is the pull back $\mathbb{R}$-algebra isomorphism defined by $\Psi^*(h)=h\circ\Psi$. The definition of the *stably S.P-$\mathcal{K}$-equivalence* among Morse families of hypersurfaces is defined to be the similar way to the definition of the stably P-$\mathcal{R}^+$-equivalence among Morse families of functions. The following is the key lemma of the theory of Legendrian singularities (cf. [1, 11, 33]).

Lemma 3.2 (Zakalyukin) *Let $F:(\mathbb{R}^k\times\mathbb{R}^m,0)\to(\mathbb{R},0)$ and $G:(\mathbb{R}^{k'}\times\mathbb{R}^m,0)\to(\mathbb{R},0)$ be Morse families of hypersurfaces. Then $(\mathcal{L}_F(\Sigma_*(F)),p)=(\mathcal{L}_G(\Sigma_*(G)),p)$ if and only if F and G are stably S.P-$\mathcal{K}$-equivalent.*

Let $F:(\mathbb{R}^k\times\mathbb{R}^m,0)\to(\mathbb{R},0)$ be a Morse family of hypersurfaces and $\Phi:(\mathbb{R}^m,0)\longrightarrow(\mathbb{R}^m,0)$ a diffeomorphism germ. We define $\Phi^*F:(\mathbb{R}^k\times\mathbb{R}^m,0)\to(\mathbb{R},0)$ by $\Phi^*F(q,x)=F(q,\Phi(x))$. Then we have $(1_{\mathbb{R}^q}\times\Phi)(\Sigma_*(\Phi^*F))=\Sigma_*(F)$ and

$$\mathscr{L}_{\Phi^*F}(\Sigma_*(\Phi^*F))=\left\{\left(x,\left[\left(\frac{\partial F}{\partial x}(q,\Phi(x))\right)\circ d\Phi_x\right]\right)\ \Big|\ (q,\Phi(x))\in\Sigma_*(F)\right\},$$

so that $\widehat{\Phi}(\mathscr{L}_{\Phi^*F}(\Sigma_*(\Phi^*F)))=\mathscr{L}_F(\Sigma_*(F))$ as set germs.

Proposition 3.3 *Let $F:(\mathbb{R}^k\times\mathbb{R}^m,0)\to(\mathbb{R},0)$ and $G:(\mathbb{R}^{k'}\times\mathbb{R}^m,0)\to(\mathbb{R},0)$ be Morse families of hypersrufaces. For a diffeomorphism germ $\Phi:(\mathbb{R}^m,0)\longrightarrow(\mathbb{R}^m,0)$, $\widehat{\Phi}(\mathcal{L}_G(\Sigma_*(G)))=\mathcal{L}_F(\Sigma_*(F))$ if and only if Φ^*F and G are stably S.P-$\mathcal{K}$-equivalent.*

Proof. Since $\widehat{\Phi}(\mathscr{L}_{\Phi^*F}(\Sigma_*(\Phi^*F)))=\mathscr{L}_F(\Sigma_*(F))$, we have $\mathscr{L}_{\Phi^*F}(\Sigma_*(\Phi^*F))=\mathcal{L}_G(\Sigma_*(G))$. By Lemma 3.2, the assertion holds. □

We say that $\mathcal{L}_F(\Sigma_*(F))$ and $\mathcal{L}_G(\Sigma_*(G))$ are *Legendrian equivalent* if there exists a diffeomorphism germ $\Phi:(\mathbb{R}^m,0)\longrightarrow(\mathbb{R}^m,0)$ such that the condition in the above proposition holds. By Lemma 3.1, under the generic condition on F and G, $\Phi(W(\mathcal{L}_G(\Sigma_*(G))))=W(\mathcal{L}_F(\Sigma_*(F)))$ if and only if $\widehat{\Phi}(\mathcal{L}_G(\Sigma_*(G)))=\mathcal{L}_F(\Sigma_*(F))$ for a diffeomorphism germ $\Phi:(\mathbb{R}^m,0)\longrightarrow(\mathbb{R}^m,0)$.

We now consider the case when $m=n+1$ and distinguish space and time coordinates, so that we denote that $\mathbb{R}^{n+1}=\mathbb{R}^n\times\mathbb{R}$ and coordinates are denoted by $(x,t)=(x_1,\dots,x_n,t)\in\mathbb{R}^n\times\mathbb{R}$. Then we consider the projective cotangent bundle $\overline{\pi}:PT^*(\mathbb{R}^n\times\mathbb{R})\to\mathbb{R}^n\times\mathbb{R}$. Because of the trivialization $PT^*(\mathbb{R}^n\times\mathbb{R})\cong(\mathbb{R}^n\times\mathbb{R})\times P((\mathbb{R}^n\times\mathbb{R})^*)$, we have homogeneous coordinates $((x_1,\dots,x_n,t),[\xi_1:\cdots:\xi_n:\tau])$. We remark that $PT^*(\mathbb{R}^n\times\mathbb{R})$ is a fiber-wise compactification of the 1-jet space as follows: We consider an affine open subset $U_\tau=\{((x,t),[\xi:\tau])|\tau\neq 0\}$ of $PT^*(\mathbb{R}^n\times\mathbb{R})$. For any $((x,t),[\xi:\tau])\in U_\tau$, we have

$$((x_1,\dots,x_n,t),[\xi_1:\cdots:\xi_n:\tau])=((x_1,\dots,x_n,t),[-(\xi_1/\tau):\cdots:-(\xi_n/\tau):-1]),$$

so that we may adopt the corresponding *affine coordinates* $((x_1,\dots,x_n,t),(p_1,\dots,p_n))$, where $p_i = -\xi_i/\tau$. On U_τ we can easily show that $\theta^{-1}(0) = K|_{U_\tau}$, where $\theta = dt - \sum_{i=1}^n p_i dx_i$. This means that U_τ can be identified with the 1-jet space which is denoted by $J^1_{GA}(\mathbb{R}^n,\mathbb{R}) \subset PT^*(\mathbb{R}^n\times\mathbb{R})$. We call the above coordinates *a system of graph-like affine coordinates.* Throughout this paper, we use this identification.

For a Legendrian submanifold $i : L \subset PT^*(\mathbb{R}^n\times\mathbb{R})$, the corresponding wave front $\overline{\pi}\circ i(L) = W(L)$ is called a *big wave front.* We call

$$W_t(L) = \pi_1(\pi_2^{-1}(t)\cap W(L)) \quad (t\in\mathbb{R})$$

a *momentary front* (or, a *small front*) for each $t \in (\mathbb{R},0)$, where $\pi_1 : \mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^n$ and $\pi_2 : \mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$ are the canonical projections defined by $\pi_1(x,t) = x$ and $\pi_2(x,t) = t$ respectively. In this sense, we call L a *big Legendrian submanifold.* We say that a point $p \in L$ is a *space-singular point* if $\operatorname{rank} d(\pi_1\circ\overline{\pi}|_L)_p < n$ and a *time-singular point* if $\operatorname{rank} d(\pi_2\circ\overline{\pi}|_L)_p = 0$, respectively. By definition, if $p\in L$ is a Legendrian singular point, then it is a space-singular point of L. Even if we have no Legendrian singular points, we have space-singular points. In this case we have the following lemma.

Lemma 3.4 *Let $i : L \subset PT^*(\mathbb{R}^n\times\mathbb{R})$ be a big Legendrian submanifold without Legendrian singular points. If $p\in L$ is a space-singular point of L, then p is not a time-singular point of L.*

Proof. By the assumption, $\overline{\pi}|_L$ is an immersion. For any $v\in T_pL$, there exists $X_v \in T_{\overline{\pi}(p)}(\mathbb{R}^n\times\{0\})$ and $Y_v\in T_{\overline{\pi}(p)}(\{0\}\times\mathbb{R})$ such that $d(\overline{\pi}|_L)_p(v) = X_v + Y_v$. If $\operatorname{rank} d(\pi_2\circ\overline{\pi}|_L)_p = 0$, then $d(\overline{\pi}|_L)_p(v) = X_v$ for any $v\in T_pL$. Since p is a space-singular point of L, there exits $v\in T_pL$ such that $X_v = 0$, so that $d(\overline{\pi}|_L)_p(v) = 0$. This contradicts to the fact that $\overline{\pi}|_L$ is an immersion. □

The *discriminant of the family $W_t(L)$* is defined as the image of singular points of $\pi_1|_{W(L)}$. In the general case, the discriminant consists of three components: *the caustic* $C_L = \pi_1(\Sigma(W(L)))$, where $\Sigma(W(L))$ is the set of singular points of $W(L)$ (i.e, the critical value set of the Legendrian mapping $\overline{\pi}|_L$). *the Maxwell stratified set* M_L, the projection of self intersection points of $W(L)$; and also of the critical value set Δ_L of $\pi_1|_{W(L)\setminus\Sigma(W(L))}$. In [24, 36], it has been written that Δ_L is the *envelope of the family of momentary fronts.* However, we remark that Δ_L is not necessary the envelope of the family of the projection of smooth momentary fronts $\overline{\pi}(W_t(L))$. There is a case that $\pi_2^{-1}(t)\cap W(L)$ is non-singular but $\pi_1|_{\pi_2^{-1}(t)\cap W(L)}$ has singularities, so that Δ_L is the set of critical values of the family of mapping $\pi_1|_{\pi_2^{-1}(t)\cap W(L)}$ for smooth $\pi_2^{-1}(t)\cap W(L)$ (cf., §5.2).

For any Legendrian submanifold germ $i : (L,p_0) \subset (PT^*(\mathbb{R}^n\times\mathbb{R}),p_0)$, there exists a generating family. Let $\mathcal{F} : (\mathbb{R}^k\times(\mathbb{R}^n\times\mathbb{R}),0)\to(\mathbb{R},0)$ be a Morse family of hypersurfaces. In this case, we call $\mathcal{F}$ a *big Morse family of hypersurfaces.* Then $\Sigma_*(\mathcal{F}) = \Delta^*(\mathcal{F})^{-1}(0)$ is a smooth n-dimensional submanifold germ. By the previous arguments, we have a big Legendrian submanifold $\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ where

$$\mathcal{L}_{\mathcal{F}}(q,x,t) = \left(x,t,\left[\frac{\partial\mathcal{F}}{\partial x}(q,x,t) : \frac{\partial\mathcal{F}}{\partial t}(q,x,t)\right]\right),$$

and

$$\left[\frac{\partial\mathcal{F}}{\partial x}(q,x,t) : \frac{\partial\mathcal{F}}{\partial t}(q,x,t)\right] = \left[\frac{\partial\mathcal{F}}{\partial x_1}(q,x,t) : \cdots : \frac{\partial\mathcal{F}}{\partial x_n}(q,x,t) : \frac{\partial\mathcal{F}}{\partial t}(q,x,t)\right].$$

We now consider an equivalence relation among Legendrian submanifolds which preserves the discriminant of families of momentary fronts. The following equivalence relation among big Legendrian submanifold germs has been independently introduced in [16, 36] for different purposes: Let $i : (L, p_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0)$ and $i' : (L', p_0') \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0')$ be big Legendrian submanifold germs. We say that i and i' are *strictly parametrized*$^+$ *Legendrian equivalent* (or, briefly *$S.P^+$-Legendrian equivalent*) if there exists a diffeomorphism germs $\Phi : (\mathbb{R}^n \times \mathbb{R}, \overline{\pi}(p_0)) \to (\mathbb{R}^n \times \mathbb{R}, \overline{\pi}(p_0'))$ of the form $\Phi(x,t) = (\phi_1(x), t + \alpha(x))$ such that $\widehat{\Phi}(L) = L'$ as set germs, where $\widehat{\Phi} : (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0) \to (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0')$ is the unique contact lift of Φ. We can also define the notion of stability of Legendrian submanifold germs with respect to $S.P^+$-Legendrian equivalence which is analogous to the stability of Lagrangian submanifold germs with respect to Lagrangian equivalence (cf. [1, Part III]). We investigate the $S.P^+$-Legendrian equivalence by using the notion of generating families of Legendrian submanifold germs. Let $\overline{f}, \overline{g} : (\mathbb{R}^k \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ be function germs. Remember that $\overline{f}$ and $\overline{g}$ are S.P-$\mathcal{K}$-equivalent if there exists a diffeomorphism germ $\Phi : (\mathbb{R}^k \times \mathbb{R}, 0) \to (\mathbb{R}^k \times \mathbb{R}, 0)$ of the form $\Phi(q,t) = (\phi(q,t), t)$ such that $\langle \overline{f} \circ \Phi \rangle_{\mathcal{E}_{k+1}} = \langle \overline{g} \rangle_{\mathcal{E}_{k+1}}$. Let $\mathcal{F}, \mathcal{G} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ be function germs. We say that $\mathcal{F}$ and $\mathcal{G}$ are *space-$S.P^+$-$\mathcal{K}$-equivalent* (or, briefly, *s-$S.P^+$-$\mathcal{K}$-equivalent*) if there exists a diffeomorphism germ $\Psi : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0)$ of the form $\Psi(q,x,t) = (\phi(q,x,t), \phi_1(x), t + \alpha(x))$ such that $\langle F \circ \Psi \rangle_{\mathcal{E}_{k+n+1}} = \langle G \rangle_{\mathcal{E}_{k+n+1}}$. The notion of $S.P^+$-$\mathcal{K}$-versal deformation plays an important role for our purpose. We define the extended tangent space of $\overline{f} : (\mathbb{R}^k \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ relative to $S.P^+$-$\mathcal{K}$ by

$$T_e(S.P^+\text{-}\mathcal{K})(\overline{f}) = \left\langle \frac{\partial \overline{f}}{\partial q_1}, \dots, \frac{\partial \overline{f}}{\partial q_k}, \overline{f} \right\rangle_{\mathcal{E}_{k+1}} + \left\langle \frac{\partial \overline{f}}{\partial t} \right\rangle_{\mathbb{R}}.$$

Then we say that F is *infinitesimally $S.P^+$-$\mathcal{K}$-versal* deformation of $\overline{f} = F|_{\mathbb{R}^k \times \{0\} \times \mathbb{R}}$ if it satisfies

$$\mathcal{E}_{k+1} = T_e(S.P^+\text{-}\mathcal{K})(\overline{f}) + \left\langle \frac{\partial \mathcal{F}}{\partial x_1}|_{\mathbb{R}^k \times \{0\} \times \mathbb{R}}, \dots, \frac{\partial \mathcal{F}}{\partial x_n}|_{\mathbb{R}^k \times \{0\} \times \mathbb{R}} \right\rangle_{\mathbb{R}}.$$

Theorem 3.5 [16, 36] *Let $\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ and $\mathcal{G} : (\mathbb{R}^{k'} \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ be big Morse families of hypersurfaces. Then*
(1) $\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ and $\mathcal{L}_{\mathcal{G}}(\Sigma_*(\mathcal{G}))$ are $S.P^+$-Legendrian equivalent if and only if $\mathcal{F}$ and $\mathcal{G}$ are stably s-$S.P^+$-$\mathcal{K}$-equivalent.
(2) $\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ is $S.P^+$-Legendre stable if and only if $\mathcal{F}$ is an infinitesimally $S.P^+$-$\mathcal{K}$-versal deformation of $\overline{f} = \mathcal{F}|_{\mathbb{R}^k \times \{0\} \times \mathbb{R}}$.

Proof. By definition, $\mathcal{F}$ and $\mathcal{G}$ are stably *s-$S.P^+$-$\mathcal{K}$*-equivalent if there exists a diffeomorphism germ $\Phi : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$ of the form $\Phi(x,t) = (\phi_1(x), t + \alpha(x))$ such that $\Phi^* F$ and G are stably $S.P$-$\mathcal{K}$-equivalent. By Proposition 3.3, we have the assertion (1). For the proof of the assertion (2), we need some more preparations, so that we omit it. We only remark here that the proof is analogous to the proof of [1, Theorem in §21.4]. □

The assumption in Proposition 3.1 is a generic condition for i, i'. Especially, if i and i' are $S.P^+$-Legendre stable, then these satisfy the assumption. Concerning the discriminant and the bifurcation of small fronts, we define the following equivalence relation among big wave front germs. Let $i : (L, p_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0)$ and $i' : (L', p'_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p'_0)$ be big Legendrian submanifold germs. We say that $W(L)$ and $W(L')$ are *$S.P^+$-diffeomorphic* if there exists a diffeomorphism germ $\Phi : (\mathbb{R}^n \times \mathbb{R}, \overline{\pi}(p_0)) \to (\mathbb{R}^n \times \mathbb{R}, \overline{\pi}(p'_0))$ of the form $\Phi(x,t) = (\phi_1(x), t + \alpha(x))$ such that $\Phi(W(L)) = W(L')$ as set germs. We remark that the $S.P^+$-diffeomorphism among big wave front germs preserves the diffeomorphism types the discriminants [36]. By Proposition 3.1, we have the following proposition.

Proposition 3.6 *Let $i : (L, p_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0)$ and $i' : (L', p'_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p'_0)$ be big Legendrian submanifold germs such that regular sets of $\overline{\pi} \circ i, \overline{\pi} \circ i'$ are dense respectively. Then i and i' are $S.P^+$-Legendrian equivalent if and only if $(W(L), \overline{\pi}(p_0))$ and $(W(L'), \overline{\pi}(p'_0))$ are $S.P^+$-diffeomorphic.*

Remark 3.7 If we consider a diffeomorphism germ $\Phi : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$ of the form $\Phi(x,t) = (\phi_1(x,t), \phi_2(t))$, we can define a *time-Legendrian equivalence* among big Legendrian submanifold germs. We can also define a *time-P-$\mathcal{K}$-equivalence* among big Morse families of hypersurfaces. By the similar arguments as the above paragraphs, we can show that these equivalence relations describe the bifurcations of momentary fronts of big Legendrian submanifolds. In [35] Zakalyukin classified generic big Legendrian submanifold germs by the time-Legendrian equivalence. The natural equivalence relation among big Legendrian submanifold germs are induced by diffeomorphism germs $\Phi : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$ of the form $\Phi(x,t) = (\phi_1(x), \phi_2(t))$. This equivalence relation classifies both the discriminants and the bifurcations of small fronts of big fronts. However, it induces a equivalence relation among divergent diagrams $(\mathbb{R}^n, 0) \leftarrow (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$, so that it is almost impossible to have a classification by this equivalence relation. Here, we remark that the corresponding group of the diffeomorphisms is not a geometric subgroup of $\mathcal{A}$ and $\mathcal{K}$ in the sense of Damon[8]. Moreover, if we consider a diffeomorphism germ $\Phi : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$ of the form $\Phi(x,t) = (\phi_1(x), t)$, we have a stronger equivalence relation among big Legendrian submanifolds, which is called an *$S.P$-Legendrian equivalence*. Although this equivalence relation gets rid of the difficulty for the above equivalence relation, there appeared function modulus for generic classifications in very low dimensions. In order to ignore the function moduli, we introduced the $S.P^+$-Legnedrian equivalence. If we have a generic classification of big Legendrian submanifold germs by the $S.P^+$-Legendrian equivalence, we have a classification by the $S.P$-Legendrian equivalence modulo function modulus. See [16, 36] for details.

On the other hand, we can also define a *space-Legendrian equivalence* among big Legendrian submanifold germs. According to the above paragraphs, we use a diffeomorphism germ $\Phi : (\mathbb{R}^n \times \mathbb{R}, 0) \longrightarrow (\mathbb{R}^n \times \mathbb{R}, 0)$ of the form $\Phi(x,t) = (\phi_1(x), \phi_2(x,t))$. The corresponding equivalence among big Morse families of hypersurfaces is the *space-P-$\mathcal{K}$-equivalence* which is analogous to the above definitions (cf., [14]). Recently, we discover an application of this equivalence relation to the geometry of world sheets in Lorentz-Minkowski space. See [27] for details.

4 Graph-like Legendrian unfoldings

In this section we explain the theory of graph-like Legendrian unfoldings and and wave front propagations. Graph-like Legendrian unfoldings belong to a special class of big Legendrian submanifolds. A big Legendrian submanifold $i : L \subset PT^*(\mathbb{R}^n \times \mathbb{R})$ is said to be a *graph-like Legendrian unfolding* if $L \subset J^1_{GA}(\mathbb{R}^n, \mathbb{R})$. We call $W(L) = \overline{\pi}(L)$ a *graph-like wave front* of L, where $\overline{\pi} : J^1_{GA}(\mathbb{R}^n, \mathbb{R}) \longrightarrow \mathbb{R}^n \times \mathbb{R}$ is the canonical projection. We define a mapping $\Pi : J^1_{GA}(\mathbb{R}^n, \mathbb{R}) \longrightarrow T^*\mathbb{R}^n$ by $\Pi(x, t, p) = (x, p)$, where $(x, t, p) = (x_1, \dots, x_n, t, p_1, \dots, p_n)$ and the canonical contact form on $J^1_{GA}(\mathbb{R}^n, \mathbb{R})$ is given by $\theta = dt - \sum_{i=1}^n p_i dx_i$. Here, $T^*\mathbb{R}^n$ is a symplectic manifold with the canonical symplectic structure $\omega = \sum_{i=1}^n dp_i \wedge dx_i$ (cf. [1]). We have the following proposition.

Proposition 4.1 ([24]) *For a graph-like Legendrian unfolding $L \subset J^1_{GA}(\mathbb{R}^n, \mathbb{R})$, $z \in L$ is a singular point of $\overline{\pi}|_L : L \longrightarrow \mathbb{R}^n \times \mathbb{R}$ if and only if it is a singular point of $\pi_1 \circ \overline{\pi}|_L : L \longrightarrow \mathbb{R}^n$. Moreover, $\Pi|_L : L \longrightarrow T^*\mathbb{R}^n$ is immersive, so that $\Pi(L)$ is a Lagrangian submanifold in $T^*\mathbb{R}^n$.*

Proof. Let $z \in L$ be a singular point of $\pi_1 \circ \overline{\pi}|_L$. Then there exists a non-zero tangent vector $\boldsymbol{v} \in T_zL$ such that $d(\pi_1 \circ \overline{\pi}|_L)_z(\boldsymbol{v}) = 0$. For the canonical coordinate (x, t, p) of $J^1_{GA}(\mathbb{R}^n, \mathbb{R})$, we have

$$\boldsymbol{v} = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i} + \beta \frac{\partial}{\partial t} + \sum_{j=1}^n \gamma_j \frac{\partial}{\partial p_j}$$

for some real numbers $\alpha_i, \beta, \gamma_j \in \mathbb{R}$. By the assumption, we have $\alpha_i = 0$ $(i = 1, \dots, n)$. Since L is a Legendrian submanifold in $J^1_{GA}(\mathbb{R}^n, \mathbb{R})$, we have $0 = \theta(\boldsymbol{v}) = \beta - \sum_{i=1}^n \gamma_i \alpha_i = \beta$. Therefore, we have

$$d\overline{\pi}(\boldsymbol{v}) = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i} + \beta \frac{\partial}{\partial t} = \mathbf{0}.$$

This means that $z \in L$ is a singular point of $\overline{\pi}|_L$. The converse assertion holds by definition.

We consider a vector $\boldsymbol{v} \in T_zL$ such that $d\Pi_z(\boldsymbol{v}) = \mathbf{0}$. By the similar reason to the above case, we have $\boldsymbol{v} = \mathbf{0}$. This means that $\Pi|_L$ is immersive. Since L is a Legendrian submanifold in $J^1_{GA}(\mathbb{R}^n, \mathbb{R})$, we have

$$\omega|_{\Pi(L)} = (\Pi|_L)^*\omega = \Pi^*\omega|_L = d\theta|_L = d(\theta|_L) = 0.$$

This completes the proof. □

We have the following corollary of Proposition 4.1.

Corollary 4.2 *For a graph-like Legendrian unfolding $L \subset J^1_{GA}(\mathbb{R}^n, \mathbb{R})$, Δ_L is the empty set, so that the discriminant of the family of small fronts is $C_L \cup M_L$.*

Since L is a big Legendrian submanifold in $PT^*(\mathbb{R}^n \times \mathbb{R})$, it has a generating family $\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ at least locally. Since $L \subset J^1_{GA}(\mathbb{R}^n, \mathbb{R}) = U_\tau \subset PT^*(\mathbb{R}^n \times \mathbb{R})$, it satisfies the condition $(\partial\mathcal{F}/\partial t)(0) \neq 0$. Let $\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ be a big Morse family of hypersurfaces. We say that $\mathcal{F}$ is a *graph-like Morse family of hypersurfaces* if $(\partial\mathcal{F}/\partial t)(0) \neq 0$. It is easy to show that the corresponding big Legendrian submanifold germ is a graph-like Legendrian unfolding. Of course, all graph-like Legendrian unfolding germs can be constructed by the above way. We say that $\mathcal{F}$ is a *graph-like generating family* of $\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$. We remark that the notion of graph-like Legendrian unfoldings and corresponding generating families have been introduced in [15] to describe the perestroikas of wave fronts given as the level surfaces of the solution for the eikonal equation given by a general Hamiltonian function. In this case, there is an additional condition, that is, $\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ is *non-degenerate* if $\mathcal{F}$ satisfies the conditions $(\partial \mathcal{F}/\partial t)(0) \neq 0$ and $\Delta^*\mathcal{F}|_{\mathbb{R}^k \times \mathbb{R}^n \times \{0\}}$ is a submersion germ. We call such a generating family $\mathcal{F}$ a *non-degenerate graph-like generating family* and corresponding graph-like Legendrian unfolding a *non-degenarate graph-like Legendrian unfolding* of $\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$. We have the following proposition.

Proposition 4.3 *Let $\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ be a graph-like Morse family of hypersurfaces. Then $\mathcal{F}$ is non-degenerate if and only if $\pi_2 \circ \overline{\pi}|_{\mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))}$ is submersive.*

Proof. By the definition of $\mathscr{L}_{\mathcal{F}}$, we have $\pi_2 \circ \overline{\pi}|_{\mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))} = \pi_2 \circ \pi_{n+1}|_{\Sigma_*(\mathcal{F})}$, where $\pi_{n+1} : \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n \times \mathbb{R}$ is the canonical projection. Since $\Sigma_*(\mathcal{F}) = \Delta^*(\mathcal{F})^{-1}(0) \subset (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0)$, $\pi_2 \circ \pi_{n+1}|_{\Sigma_*(\mathcal{F})}$ is submersive if and only if

$$\operatorname{rank}\left(\frac{\partial \Delta^*(\mathcal{F})}{\partial q}(0), \frac{\partial \Delta^*(\mathcal{F})}{\partial x}(0)\right) = k+1.$$

The last condition is equivalent to the condition that

$$\Delta^*(\mathcal{F}|_{\mathbb{R}^k \times \mathbb{R}^n \times \{0\}}) : (\mathbb{R}^k \times \mathbb{R}^n \times \{0\}, 0) \longrightarrow (\mathbb{R} \times \mathbb{R}^k, 0)$$

is non-singular. This completes the proof. □

We say that a graph-like Legendrian unfolding $L \subset J^1_{GA}(\mathbb{R}^n, \mathbb{R})$ is *non-degenerate* if $\pi_2 \circ \overline{\pi}|L$ is submersive. The notion of graph-like Legendrian unfolding was firstly introduced in [15]. In the first time, it was assumed the non-degeneracy. However, during the last two decades, we have clarified the situation and now it is defined as non-degenerate graph-like Legendrian unfoldings.

We can reduce more strict form of graph-like generating families as follows: Let $\mathcal{F}$ be a graph-like Morse family of hypersurfaces. By the implicit function theorem, there exists a function $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ such that $\langle \mathcal{F}(q, x, t)\rangle_{\mathcal{E}_{k+n+1}} = \langle F(q, x) - t\rangle_{\mathcal{E}_{k+n+1}}$. Then we have the following proposition.

Proposition 4.4 *Let $\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ and $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be function germs such that $\langle \mathcal{F}(q, x, t)\rangle_{\mathcal{E}_{k+n+1}} = \langle F(q, x) - t\rangle_{\mathcal{E}_{k+n+1}}$. Then $\mathcal{F}$ is a graph-like Morse family of hypersurfaces if and only if F is a Morse family of functions.*

Proof. By the assumption, there exists $\lambda(q, x, t) \in \mathcal{E}_{k+n+1}$ such that $\lambda(0) \neq 0$ and

$$\mathcal{F}(q, x, t) = \lambda(q, x, t)(F(q, x) - t).$$

Since $\partial \mathcal{F}/\partial q_i = \partial \lambda/\partial q_i(F - t) + \lambda \partial F/\partial q_i$, we have

$$\Delta^*(\mathcal{F}) = (\mathcal{F}, d_1\mathcal{F}) = \left(\lambda(F - t), \frac{\partial \lambda}{\partial q}(F - t) + \lambda \frac{\partial F}{\partial q}\right),$$

where

$$\frac{\partial \lambda}{\partial q}(F-t)+\lambda\frac{\partial F}{\partial q}=\left(\frac{\partial \lambda}{\partial q_1}(F-t)+\lambda\frac{\partial F}{\partial q_1},\dots,\frac{\partial \lambda}{\partial q_k}(F-t)+\lambda\frac{\partial F}{\partial q_k}\right).$$

By straightforward calculations, the Jacobian matrix of $\Delta^*(\mathcal{F})(0)$ is

$$J_{\Delta^*(\mathcal{F})}(0)=\begin{pmatrix} 0 & \lambda(0)\frac{\partial F}{\partial x}(0) & -\lambda(0) \\ \lambda(0)\frac{\partial^2 F}{\partial q^2}(0) & \lambda(0)\frac{\partial^2 F}{\partial x \partial q}(0) & 0 \end{pmatrix}$$

We remark that the Jacobi matrix of ΔF is given by $J_{\Delta F}=(\partial^2 F/\partial q^2\ \partial^2 F/\partial x\partial q)$. Therefore, rank $J_{\Delta^*(\mathcal{F})}(0)=k+1$ if and only if rank $J_{\Delta F}(0)=k$. This completes the proof. $\square$

We now consider the case $\mathcal{F}(q,x,t)=\lambda(q,x,t)(F(q,x)-t)$. In this case,

$$\Sigma_*(\mathcal{F})=\{(q,x,F(q,x))\in(\mathbb{R}^k\times(\mathbb{R}^n\times\mathbb{R}),0)\mid (q,x)\in C(F)\},$$

where $C(F)=\Delta F^{-1}(0)$. Moreover, we have the Lagrangian submanifold germ $L(F)(C(F))\subset T^*\mathbb{R}^n$, where

$$L(F)(q,x)=\left(x,\frac{\partial F}{\partial x_1}(q,x),\dots,\frac{\partial F}{\partial x_n}(q,x)\right).$$

Since $\mathcal{F}$ is a graph-like Morse family of hypersurfaces, we have a big Legendrian submanifold germ $\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))\subset J^1_{GA}(\mathbb{R}^n,\mathbb{R})$, where $\mathcal{L}_{\mathcal{F}}:(\Sigma_*(\mathcal{F}),0)\to J^1_{GA}(\mathbb{R}^n,\mathbb{R})$ defined by

$$\mathcal{L}_{\mathcal{F}}(q,x,t)=\left(x,t,-\frac{\frac{\partial \mathcal{F}}{\partial x_1}(q,x,t)}{\frac{\partial \mathcal{F}}{\partial t}(q,x,t)},\dots,-\frac{\frac{\partial \mathcal{F}}{\partial x_n}(q,x,t)}{\frac{\partial \mathcal{F}}{\partial t}(q,x,t)},\right)\in J^1_{GA}(\mathbb{R}^n,\mathbb{R})\cong T^*\mathbb{R}^n\times\mathbb{R}.$$

We also define a map $\mathfrak{L}_F:(C(F),0)\to J^1_{GA}(\mathbb{R}^n,\mathbb{R})$ by

$$\mathfrak{L}_F(q,x)=\left(x,F(q,x),\frac{\partial F}{\partial x_1}(q,x),\dots,\frac{\partial F}{\partial x_n}(q,x)\right).$$

Since $\partial\mathcal{F}/\partial x_i=\partial\lambda/\partial x_i(F-t)+\lambda\partial F/\partial x_i$ and $\partial\mathcal{F}/\partial t=\partial\lambda/\partial t(F-t)-\lambda$, we have $\partial\mathcal{F}/\partial x_i(q,x,t)=\lambda(q,x,t)\partial F/\partial x_i(q,x,t)$ and $\partial\mathcal{F}/\partial t(q,x,t)=-\lambda(q,x,t)$ for $(q,x,t)\in\Sigma_*(\mathcal{F})$. It follows that $\mathfrak{L}_F(C(F))=\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$. By definition, we have $\Pi(\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F})))=\Pi(\mathfrak{L}_F(C(F)))=L(F)(C(F))$. The graph-like wave front of the graph-like Legendrian unfolding $\mathfrak{L}_F(C(F))=\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ is the graph of $F|_{C(F)}$. This is the reason why we call it the graph-like Legendrian unfolding. For a non-degenrate graph-like Morse family of hypersurfaces, we have the following proposition.

Proposition 4.5 *Under the same notations of Proposition 4.4, $\mathcal{F}$ is a non-degenerate graph-like Morse family of hypersurfaces if and only if F is a Morse family of hypersurfaces. In this case, F is also a Morse family of functions such that*

$$\left(\frac{\partial F}{\partial x_1}(0),\dots,\frac{\partial F}{\partial x_n}(0)\right)\neq\mathbf{0}.$$

Proof. By exactly the same calculations as those in the proof of Proposition 4.4, the Jacobi matrix of $\Delta^*(\mathcal{F}|_{\mathbb{R}^k\times\mathbb{R}^n\times\{0\}})$ is

$$J_{\Delta^*(\mathcal{F}|_{\mathbb{R}^k\times\mathbb{R}^n\times\{0\}})}(0) = \begin{pmatrix} 0 & \lambda(0)\frac{\partial F}{\partial x}(0)) \\ \lambda(0)\frac{\partial^2 F}{\partial q^2}(0) & \lambda(0)\frac{\partial^2 F}{\partial x \partial q}(0) \end{pmatrix}.$$

On the other hand, the Jacobi matrix of $\Delta^*(F)$ is

$$J_{\Delta^*(F)}(0) = \begin{pmatrix} 0 & \frac{\partial F}{\partial x}(0)) \\ \frac{\partial^2 F}{\partial q^2}(0) & \frac{\partial^2 F}{\partial x \partial q}(0) \end{pmatrix},$$

so that the first assertion holds. Moreover, $\operatorname{rank} J_{\Delta^*(\mathcal{F}|_{\mathbb{R}^k\times\mathbb{R}^n\times\{0\}})}(0) = k+1$ implies $\operatorname{rank} J_{\Delta F}(0) = k$ and $\partial F/\partial x(0) \neq \mathbf{0}$. This completes the proof. □

The family of momentary fronts is $W_t(L) = \pi_1(\pi_2^{-1}(t) \cap W(L))$. We define $L_t = L \cap (\pi_2 \circ \overline{\pi})^{-1}(t) = L \cap (T^*\mathbb{R}^n \times \{t\})$ under the canonical identification $J^1_{GA}(\mathbb{R}^n, \mathbb{R}) \cong T^*\mathbb{R}^n \times \mathbb{R}$. Then $\Pi(L) \subset T^*\mathbb{R}^n$ and $\widetilde{\pi} \circ \Pi(L_t) \subset PT^*\mathbb{R}^n$, where $\widetilde{\pi} : T^*\mathbb{R}^n \longrightarrow PT^*(\mathbb{R}^n)$ is the canonical projection. We also have the canonical projections $\varpi : T^*\mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $\overline{\varpi} : PT^*\mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $\pi_1 \circ \overline{\pi} = \varpi \circ \Pi$ and $\overline{\varpi} \circ \widetilde{\pi} = \varpi$. We have the following proposition.

Proposition 4.6 *Let $L \subset J^1_{GA}(\mathbb{R}^n, \mathbb{R})$ be a non-degenerate graph-like Legendrian unfolding. Then $\Pi(L)$ is a Lagrangian submanifold and $\widetilde{\pi} \circ \Pi(L_t)$ is a Legendrian submanifold in $PT^*(\mathbb{R}^n)$.*

Proof. By Proposition 4.1, $\Pi(L)$ is a Lagrangian submanifold in $T^*\mathbb{R}^n$. Since L is a non-degenerate Legendrian unfolding in $J^1_{GA}(\mathbb{R}^n, \mathbb{R})$, we have a non-degenrate graph-like generating family $\mathcal{F}$ of L at least locally. This means that $L = \mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ as set germs. Since $\mathcal{F}$ is a graph-like Morse family of hypersurface, it is written as $\mathcal{F}(q, x, t) = \lambda(q, x, t)(F(q, x) - t)$. Therefore, we have $\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F})) = \mathfrak{L}_F(C(F))$. By definition, $\Pi \circ \mathfrak{L}_F(C(F)) = L(F)(C(F))$, so that F is a generating family of $\Pi(L)$ locally. By Proposition 4.5, F is also a Morse family of hypersurface, so that $\mathscr{L}_F(\Sigma_*(F))$ is a Legendrian submanifold germ in $PT^*(\mathbb{R}^n)$. Without the loss of generality, we assume that $t = 0$. Since $\Sigma_*(F) = C(F) \cap F^{-1}(0)$, $\mathscr{L}_F(\Sigma_*(F)) = \widetilde{\pi} \circ \Pi(\mathcal{L}_F(C(F)) \cap (\pi_2 \circ \overline{\pi})^{-1}(0)) = \widetilde{\pi} \circ \Pi(L_0)$. This completes the proof. □

In general, the momentary front $W_t(L)$ of a big Legendrian submanifold $L \subset PT^*(\mathbb{R}^n \times \mathbb{R})$ is not necessary a wave front of a Legendrian submanifold in the ordinary sense. However, for a non-degenerate Legendrian unfolding in $J^1_{GA}(\mathbb{R}^n, \mathbb{R})$, we have the following corollary.

Corollary 4.7 *Let $L \subset J^1_{GA}(\mathbb{R}^n, \mathbb{R})$ be a non-degenerate graph-like Legendrian unfolding. Then the momentary front $W_t(L)$ is the wave front set of the Legendrian submanifold $\widetilde{\pi} \circ \Pi(L_t) \subset PT^*(\mathbb{R}^n)$. Moreover, the caustic C_L is the caustic of the Lagrangian submanifold $\Pi(L) \subset T^*\mathbb{R}^n$. In other wards, $W_t(L) = \overline{\varpi}(\widetilde{\pi} \circ \Pi(L_t))$ and C_L is the singular value set of $\varpi|\Pi(L)$.*

Proof. By definition, we have

$$\overline{\pi}(L_t) = \overline{\pi}(L \cap (\pi_2 \circ \overline{\pi})^{-1}(t)) = W(L) \cap \pi_2^{-1}(t),$$

so that

$$W_t(L) = \pi_1(W(L) \cap \pi_2^{-1}(t)) = \pi_1 \circ \overline{\pi}(L_t) = \varpi \circ \Pi(L_t) = \overline{\varpi}(\widetilde{\pi} \circ \Pi(L_t)).$$

We remark that $\pi_1 \circ \overline{\pi} = \varpi \circ \Pi$. By Proposition 4.1, $z \in L$ is a singular point of $\overline{\pi}|_L : L \longrightarrow \mathbb{R}^n \times \mathbb{R}$ if and only if it is a singular point of $\varpi|_{\Pi(L)} : \Pi(L) \longrightarrow \mathbb{R}^n$. Therefore, the caustic C_L is the singular value set of $\varpi|_{\Pi(L)}$. □

For a graph-like Morse family of hypersurfaces $\mathcal{F}(q,x,t) = \lambda(q,x,t)(F(q,x) - t)$, $\mathcal{F}$ and $\overline{F}(q,x,t) = F(q,x) - t$ are s-$S.P^+$-$\mathcal{K}$-equivalent, so that we can use $\overline{F}(q,x,t) = F(q,x) - t$ as a graph-like Morse family. Moreover, if $\mathcal{F}$ is non-degenerate, then $F(q,x)$ is a Morse family of functions. We now suppose that $F(q,x)$ is a Morse family of functions. Consider the graph-like Morse family of hypersurfaces $\overline{F}(q,x,t) = F(q,x) - t$ which is not necessarily non-degenerate. Then we have $\mathscr{L}_{\overline{F}}(\Sigma_*(\overline{F})) = \mathfrak{L}_F(C(F))$. We also denote that $\overline{f}(q,t) = f(q) - t$ for any $f \in \mathfrak{M}_k$. We can represent the extended tangent space of $\overline{f} : (\mathbb{R}^k \times \mathbb{R}, 0) \longrightarrow (\mathbb{R}, 0)$ relative to $S.P^+$-$\mathcal{K}$ by

$$T_e(S.P^+\text{-}\mathcal{K})(\overline{f}) = \left\langle \frac{\partial f}{\partial q_1}(q), \ldots, \frac{\partial f}{\partial q_k}(q), f(q) - t \right\rangle_{\mathcal{E}_{(q,t)}} + \langle 1 \rangle_{\mathbb{R}}.$$

For a deformation $\overline{F} : (\mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}, 0) \longrightarrow (\mathbb{R}, 0)$ of $\overline{f}$, $\overline{F}$ is infinitesimally $S.P^+$-$\mathcal{K}$-versal deformation of $\overline{f}$ if and only if

$$\mathcal{E}_{(q,t)} = T_e(S.P^+\text{-}\mathcal{K})(\overline{f}) + \left\langle \frac{\partial F}{\partial x_1}|_{\mathbb{R}^k \times \{0\}}, \ldots, \frac{\partial F}{\partial x_n}|_{\mathbb{R}^k \times \{0\}} \right\rangle_{\mathbb{R}}.$$

We consider a relationship of the equivalence relations between Lagrangian submanifold germs and induced graph-like Legendrian unfoldings, that is, between Morse families of functions and big Morse families of graph-like Legendrian unfoldings. As a consequence, we give a relationship between caustics and graph-like wave fronts.

Proposition 4.8 ([24]) *If Lagrangian submanifold germs $L(F)(C(F))$, $L(G)(C(G))$ are Lagrangian equivalent, then the graph-like Legendrian unfoldings $\mathfrak{L}_F(C(F))$, $\mathfrak{L}_G(C(G))$ are $S.P^+$-Legendrian equivalent.*

Proof. By Proposition 2.1, two Lagrangian submanifold germs $L(F)(C(F))$, $L(G)(C(G))$ are Lagrangian equivalent if and only if F and G are stably P-$\mathcal{R}^+$-equivalent. By definition, if F and G are stably P-$\mathcal{R}^+$-equivalent, then $\overline{F}$ and $\overline{G}$ are stably s-$S.P^+$-$\mathcal{K}$-equivalent. By the assertion (1) of Theorem 3.5, $\mathfrak{L}_F(C(F))$ and $\mathfrak{L}_G(C(G))$ are $S.P^+$-Legendrian equivalent. □

Remark 4.9 The above proposition asserts that the Lagrangian equivalence is stronger equivalence relation than the $S.P^+$-Legendrian equivalence. The $S.P^+$-Legendrian equivalence relation among graph-like Legendrian unfoldings preserves both the diffeomorphism types of caustics and Maxwell stratified sets. On the other hand, if we observe the real caustics of rays, we cannot observe the structure of wave front propagations and the

Maxwell stratified sets. In this sense, there are hidden structures behind the picture of real caustics. By the above proposition, the Lagrangian equivalence preserve not only the diffeomorphism type of caustics, but also the hidden geometric structure of wave front propagations.

It seems that the converse assertion does not hold. However, we have the following proposition.

Proposition 4.10 ([25]) *Suppose that $L(F)(C(F))$ and $L(G)(C(G))$ are Lagrange stable. If the graph-like Legendrian unfoldings $\mathfrak{L}_F(C(F))$ and $\mathfrak{L}_G(C(G))$ are $S.P^+$-Legendrian equivalent, then the Lagrangian submanifold germs $L(F)(C(F))$ and $L(G)(C(G))$ are Lagrangian equivalent.*

In order to show the proposition, we need the following lemma:

Lemma 4.11 *If $\overline{f}$ and $\overline{g} : (\mathbb{R}^k \times \mathbb{R}, 0) \longrightarrow (\mathbb{R}, 0)$ are $S.P$-$\mathcal{K}$-equivalent, then f and $g : (\mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ are $\mathcal{R}$-equivalent, where $\overline{f}(q,t) = f(q) - t$ and $\overline{g}(q,t) = g(q) - t$.*

Proof. By definition of $S.P$-$\mathcal{K}$-equivalent, there exist a diffeomorphsim germ of $\Phi : (\mathbb{R}^k \times \mathbb{R}, 0) \longrightarrow (\mathbb{R}^k \times \mathbb{R}, 0)$ of the form $\Phi(q,t) = (\phi(q,t), t)$ and a non-zero function germ $\lambda : (\mathbb{R}^k \times \mathbb{R}, 0) \longrightarrow \mathbb{R}$ such that $\overline{f} = \lambda \cdot \overline{g} \circ \Phi$. Then the diffeomorphism Φ preserves the zero-level set of $\overline{f}$ and $\overline{g}$, that is, $\Phi(\overline{f}^{-1}(0)) = \overline{g}^{-1}(0)$. Since the zero-level set of $\overline{f}$ is the graph of f and the form of Φ, we have $f = g \circ \psi$, where $\psi(q) = \phi(q, f(q))$. It is easy to show that $\psi : (\mathbb{R}^k, 0) \longrightarrow (\mathbb{R}^k, 0)$ is a diffeomorphism germ. Hence f and g are $\mathcal{R}$-equivalent. □

Proof of Prorposition 4.10. By the assertion (1) of Theorem 3.5, $\overline{F}$ and $\overline{G}$ are stably S-$S.P^+$-$\mathcal{K}$-equivalent. It follows that $\overline{f}$ and $\overline{g}$ are stably $S.P$-$\mathcal{K}$-equivalent. By Lemma 4.11, f and g are stably $\mathcal{R}$-equivalent. By the uniqueness of the infinitesimally $\mathcal{R}^+$-versal unfolding (cf., [6]), F and G are stably P-$\mathcal{R}^+$-equivalent. □

By definition, the set of Legendrian singular points of graph-like Legendrian unfolding $\mathfrak{L}_F(C(F))$ coinsides with the set of singular points of $\pi \circ L(F)$. Therefore the singularities of graph-like wave fronts of $\mathfrak{L}_F(C(F))$ lie on the caustics of $L(F)$. Moreover, if Lagrangian submanifold germ $L(F)(C(F))$ is a Lagrange stable, then the regular set of $\overline{\pi} \circ \mathfrak{L}_F(C(F))$ is dense. Hence we can apply Proposition 3.1 to our situations and obtain the following theorem as a corollary of Propositions 4.8 and 4.10.

Theorem 4.12 ([25]) *Suppose that $L(F)(C(F))$ and $L(G)(C(G))$ are Lagrange stable. Then Lagrangian subamnifold germs $L(F)(C(F))$ and $L(G)(C(G))$ are Lagrangian equivalent if and only if graph-like wave fronts $W(\mathfrak{L}_F)$ and $W(\mathfrak{L}_G)$ are $S.P^+$-diffeomorphic.*

Moreover, we have the following theorem.

Theorem 4.13 ([26]) *Suppose that $\mathcal{F}(q,x,t) = \lambda(q,x,t)\langle F(q,x) - t\rangle$ is a graph-like Morse family of hypersurfaces. Then $\mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ is $S.P^+$-Legendrian stable if and only if $L(F)(C(F))$ is Lagrangian stable.*

Proof. By Proposition 4.8, if $L(F)(C(F))$ is Lagrangian stable, then $\mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ is $S.P^+$-Legendrian stable. For the converse, suppose that $\mathfrak{L}_F(C(F))$ is a $S.P^+$-Legendre stable. By the assertion (2) of Theorem 3.5, we have

$$\dim_{\mathbb{R}} \frac{\mathcal{E}_{k+1}}{\langle \frac{\partial f}{\partial q_1}(q), \dots, \frac{\partial f}{\partial q_k}(q), f(q)-t\rangle_{\mathcal{E}_{k+1}} + \langle 1\rangle_{\mathbb{R}}} < \infty.$$

It follows that $\dim_{\mathbb{R}}\mathcal{E}_k/\langle \frac{\partial f}{\partial q_1}(q), \dots, \frac{\partial f}{\partial q_k}(q), f(q)\rangle_{\mathcal{E}_k} < \infty$, namely, f is a $\mathcal{K}$-finitely determined (see the definition [9, 29]). It is a well-known result that f is a $\mathcal{K}$-finitely determined if and only if f is an $\mathcal{R}^+$-finitely determined, see [9]. Under the condition that f is an $\mathcal{R}^+$-finitely determined, F is an infinitesimally $\mathcal{R}^+$-versal deformation of f if and only if F is an $\mathcal{R}^+$-transversal deformation of f, namely, there exists a number $\ell \in \mathbb{N}$ such that

$$\mathcal{E}_k = J_f + \left\langle \frac{\partial F}{\partial x_1}|_{\mathbb{R}^k\times\{0\}}, \dots, \frac{\partial F}{\partial x_n}|_{\mathbb{R}^k\times\{0\}} \right\rangle_{\mathbb{R}} + \langle 1\rangle_{\mathbb{R}} + \mathcal{M}_k^{\ell+1}. \tag{1}$$

Hence, it is enough to show the equality (1). Let $g(q) \in \mathcal{E}_k$. Since $g(q) \in \mathcal{E}_{k+1}$, there exist $\lambda_i(q,t), \mu(q,t) \in \mathcal{E}_{k+1}$ $(i = 1, \dots, k)$ and $c, c_j \in \mathbb{R}$ $(j = 1, \dots, n)$ such that

$$g(q) = \sum_{i=1}^{k} \lambda_i(q,t)\frac{\partial f}{\partial q_i}(q) + \mu(q,t)(f(q)-t) + c + \sum_{j=1}^{n} c_j \frac{\partial F}{\partial x_j}(q,0). \tag{2}$$

Differentiating the equality (2) with respect to t, we have

$$0 = \sum_{i=1}^{k} \frac{\partial \lambda_i}{\partial t}(q,t)\frac{\partial f}{\partial q_i}(q) + \frac{\partial \mu}{\partial t}(q,t)(f(q)-t) - \mu(q,t). \tag{3}$$

We put $t = 0$ in (3), $0 = \sum_{i=1}^{k}(\partial\lambda_i/\partial t)(q,0)(\partial f/\partial q_i)(q) + (\partial\mu/\partial t)(q,0)f(q) - \mu(q,0)$. Also we put $t = 0$ in (2), then

$$\begin{aligned} g(q) &= \sum_{i=1}^{k} \lambda_i(q,0)\frac{\partial f}{\partial q_i}(q) + \mu(q,0)f(q) + c + \sum_{j=1}^{n} c_j \frac{\partial F}{\partial x_j}(q,0) \\ &= \sum_{i=1}^{k} \alpha_i(q)\frac{\partial f}{\partial q_i}(q) + \frac{\partial \mu}{\partial t}(q,0)f^2(q) + c + \sum_{j=1}^{n} c_j \frac{\partial F}{\partial x_j}(q,0), \end{aligned} \tag{4}$$

for some $\alpha_i \in \mathcal{E}_k, i = 1\dots,k$. Again differentiating (3) with respect to t and put $t = 0$, then

$$0 = \sum_{i=1}^{k} \frac{\partial^2 \lambda_i}{\partial t^2}(q,0)\frac{\partial f}{\partial q_i}(q) + \frac{\partial^2 \mu}{\partial t^2}(q,0)f(q) - 2\frac{\partial \mu}{\partial t}(q,0).$$

Hence (4) is equal to

$$\sum_{i=1}^{k} \beta_i(q)\frac{\partial f}{\partial q_i}(q) + \frac{1}{2}\frac{\partial^2 \mu}{\partial t^2}(q,0)f^3(q) + c + \sum_{j=1}^{n} c_j \frac{\partial F}{\partial x_j}(q,0),$$

for some $\beta_i \in \mathcal{E}_k, i = 1, \dots, k$. Inductively, we take ℓ-times differentiate (3) with respect to t and put $t = 0$, then we have

$$g(q) = \sum_{i=1}^{k} \gamma_i(q) \frac{\partial f}{\partial q_i}(q) + \frac{1}{\ell !} \frac{\partial^\ell \mu}{\partial t^\ell}(q,0) f^{\ell+1}(q) + c + \sum_{j=1}^{n} c_j \frac{\partial F}{\partial x_j}(q,0),$$

for some $\gamma_i \in \mathcal{E}_k, i = 1, \dots, k$. It follows that $g(q)$ is contained in the right hand of (1). This completes the proof. □

One of the consequences of the above arguments, we have the following theorem on the relation among graph-like Legendrian unfoldings and Lagrangian singularities.

Theorem 4.14 *Let $\mathcal{F} : (\mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}, 0) \longrightarrow (\mathbb{R}, 0)$ and $\mathcal{G} : (\mathbb{R}^{k'} \times \mathbb{R}^n \times \mathbb{R}, 0) \longrightarrow (\mathbb{R}, 0)$ be graph-like Morse families of hypersurface of the forms $\mathcal{F}(q,x,t) = \lambda(q,x,t)(F(q,x) - t)$ and $\mathcal{G}(q',x,t) = \mu(q',x,t)(G(q',x) - t)$ such that $\mathscr{L}_\mathcal{F}(\Sigma_*(\mathcal{F}))$ and $\mathscr{L}_\mathcal{G}(\Sigma_*(\mathcal{G}))$ are $S.P^+$-Legendrian stable. Then the following conditions are equivalent:*
(1) $\mathscr{L}_\mathcal{F}(\Sigma_*(\mathcal{F}))$ and $\mathscr{L}_\mathcal{G}(\Sigma_*(\mathcal{G}))$ are $S.P^+$-Legendrian equivalent,
(2) $\mathcal{F}$ and $\mathcal{G}$ are stably s-$S.P^+$-$\mathcal{K}$-equivalent,
(3) $\overline{f}(q,t) = F(q,0) - t$ and $\overline{g}(q',t) = G(q',0) - t$ are stably $S.P$-$\mathcal{K}$-equivalent,
(4) $f(q) = F(q,0)$ and $g(q') = G(q',0)$ are stably $\mathcal{R}$-equivalent,
(5) $F(q,x)$ and $G(q',x)$ are stably P-$\mathcal{R}^+$-equivalent,
(6) $L(F)(C(F))$ and $L(G)(C(G))$ are Lagrangian equivalent,
(7) $W(\mathscr{L}_\mathcal{F}(\Sigma_*(\mathcal{F})))$ and $W(\mathscr{L}_\mathcal{G}(\Sigma_*(\mathcal{G})))$ are $S.P^+$-diffeomorphic.

Proof. By the assertion (1) o f Theorem 3.5, the conditions (1) and (2) are equivalent. By definition, the condition (2) implies the condition (3), the condition (4) implies (3) and the condition (5) implies (4), respectively. By Lemma 4.11, the condition (3) implies the condition (4). By Theorem 2.1, the conditions (5) and (6) are equivalent. It also follows from the definition that the condition (1) implies (7). We remark that all these assertions hold without the assumptions of the $S.P^+$-Legendrian stability. In generic, the condition (7) implies the condition (1) by Proposition 3.1. Of course, it holds by Theorem 4.12 under the assumption of the $S.P^+$-Legendrian stability. By the assumption of the $S.P^+$-Legendrian stability, the graph-like Morse families of hypersurface $\mathcal{F}$ and $\mathcal{G}$ are infinitesimally $S.P^+$-$\mathcal{K}$-versal deformations of $\overline{f}$ and $\overline{g}$, respectively (cf., Theorem 3.5, (2)). By the uniqueness result of the infinitesimally $S.P^+$-$\mathcal{K}$-versal deformations, the condition (3) implies the condition (2). Moreover, by Theorem 4.13, $L(F)(C(F))$ and $L(G)(C(G))$ are Lagrangian stable. This means that F and G are infinitesimally $\mathcal{R}^+$-versal deformations of f and g, respectively. Therefore by the uniqueness results on the infinitesimally $\mathcal{R}^+$-versal deformations, the condition (4) implies the condition (5). This completes the proof. □

Remark 4.15 (1) By Theorem 4.13, the assumption of the above theorem is equivalent to the condition that $L(F)(C(F))$ and $L(G)(C(G))$ are Lagrangian stable.
(2) If $k = k'$ and $q = q'$ in the above theorem, we can remove the word "stably" in the conditions (2),(3),(4) and (5).

(3) The $S.P^+$-Legendrian stablility of $\mathscr{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ is a generic condition for $n \leq 5$.
(4) By the remark in the proof of the above theorem, the conditions (1) and (7) are equivalent in generic for a general dimension n without the assumption on the $S.P^+$-Legendrian stability. Therefore, the conditions (1),(2) and (7) are all equivalent to each other in generic. The Lagrangian equivalence (i.e., the conditions (5) and (6)) is a stronger condition than others even in generic.

5 Applications

In this section we explain some applications of the theory of wave front propagations.

5.1 Completely integrable first order ordinary differential equations

In this subsection we consider implicit first order ordinary equations. It is classically written by $F(x, y, dy/dx) = 0$. However, if we set $p = dy/dx$, then we have a surface on the 1-jet space $J^1(\mathbb{R}, \mathbb{R})$ defined by $F(x, y, p) = 0$, where we have the canonical contact form $\theta = dy - pdx$. Generically, we may assume that the surface is regular, then it has a local parametrization, so that it is an image of an immersion at least locally. An *ordinary differential equation germ* (briefly, an *ODE*) is defined to be an immersion germ $f : (\mathbb{R}^2, 0) \longrightarrow J^1(\mathbb{R}, \mathbb{R})$. We say that an ODE $f : (\mathbb{R}^2, 0) \longrightarrow J^1(\mathbb{R}, \mathbb{R})$ is *completely integrable* if there exists a submersion germ $\mu : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}, 0)$ such that $f^*\theta \in \langle d\mu \rangle_{\mathcal{E}_2}$. It follows that there exists a unique $h \in \mathcal{E}_2$ such that $f^*\theta = hd\mu$. In this case we call μ an *complete integral* of f. In [12] it has been considered a generic classification of completely integrable first order ODEs by point transformations. Let $f, g : (\mathbb{R}^2, 0) \longrightarrow J^1(\mathbb{R}, \mathbb{R}) \subset PT^*(\mathbb{R} \times \mathbb{R})$ be ODEs. We say that f, g are *equivalent* as ODEs if there exist diffeomorphism germs $\psi : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^2, 0)$ and $\Phi : (\mathbb{R} \times \mathbb{R}, 0) \longrightarrow (\mathbb{R} \times \mathbb{R}, 0)$ such that $\hat{\Phi} \circ f = g \circ \psi$. Here $\hat{\Phi}$ is the unique contact lift of Φ. The diffeomorphism germ $\Phi : (\mathbb{R} \times \mathbb{R}, 0) \longrightarrow (\mathbb{R} \times \mathbb{R}, 0)$ is traditionally called a *point transformation.* We represent f by the canonical coordinates of $J^1(\mathbb{R}, \mathbb{R})$ by $f(u_1, u_2) = (x(u_1, u_2), y(u_1, u_2), p(u_1, u_2))$. If we have a complete integral $\mu : (\mathbb{R}^2, 0) \longrightarrow \mathbb{R}$ of f, we define a immersion germ $\ell_{(\mu,f)} : (\mathbb{R}^2, 0) \longrightarrow J^1(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ by

$$\ell_{(\mu,f)}(u_1, u_2) = (\mu(u_1, u_2), x(u_1, u_2), y(u_1, u_2), h(u_1, u_2), p(u_1, u_2)).$$

Then we have $\ell^*_{(\mu,f)}\Theta = 0$, for $\Theta = dy - pdx - qdt$, where (t, x, y, q, p) is the canonical coordinate of $J^1(\mathbb{R} \times \mathbb{R}, \mathbb{R})$. Therefore, the image of $\ell_{(\mu,f)}$ is a big Legendrian submanifold germ of $J^1(\mathbb{R} \times \mathbb{R}, \mathbb{R})$. Here, we consider the parameter t as the time-parameter. Since the contact structure is defined by the contact form $\Theta = dy - pdx - qdt$, $J^1(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ is of course an affine coordinate neighbourhood of $PT^*(\mathbb{R} \times \mathbb{R} \times \mathbb{R})$ but it is not equal to $J^1_{AG}(\mathbb{R} \times \mathbb{R}, \mathbb{R}) \subset PT^*(\mathbb{R} \times \mathbb{R} \times \mathbb{R})$. The above notation induces a divergent diagram of map germs as follows:

$$\mathbb{R} \overset{\pi_1 \circ \overline{\pi} \circ \ell_{(\mu,f)}}{\longleftarrow} (\mathbb{R}^2, 0) \overset{\pi_2 \circ \overline{\pi} \circ \ell_{(\mu,f)}}{\longrightarrow} (\mathbb{R} \times \mathbb{R}, 0),$$

where $\overline{\pi} : J^1(\mathbb{R} \times \mathbb{R}, \mathbb{R}) \longrightarrow \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ is $\overline{\pi}(t, x, y, q, p) = (t, x, y)$, $\pi_1 : (\mathbb{R} \times \mathbb{R} \times \mathbb{R}, 0) \longrightarrow \mathbb{R}$ is $\pi_1(t, x, y) = t$ and $\pi_2 : (\mathbb{R} \times \mathbb{R} \times \mathbb{R}, 0) \longrightarrow \mathbb{R} \times \mathbb{R}$ is $\pi_2(t, x, y) = (x, y)$. Actually, we have

$\pi_1 \circ \overline{\pi} \circ \ell_{(\mu,f)} = \mu$ and $\pi_2 \circ \overline{\pi} \circ \ell_{(\mu,f)} = \widehat{\pi} \circ f$, where $\widehat{\pi} : J^1(\mathbb{R},\mathbb{R}) \longrightarrow \mathbb{R}\times\mathbb{R}$ is the canonical projection $\widehat{\pi}(x,y,p) = (x,y)$. The space of completely integrable ODEs is identified with the space of big Legendrian submanifold with the restriction of the $\pi_1 \circ \overline{\pi}$-projection is non-singular. For a divergent diagram

$$\mathbb{R} \xleftarrow{\mu} (\mathbb{R}^2, 0) \xrightarrow{g} (\mathbb{R}\times\mathbb{R}, 0),$$

we say that (μ, g) is an *integral diagram* if there exist an immersion germ $f : (\mathbb{R}^2, 0) \longrightarrow J^1(\mathbb{R},\mathbb{R})$ and a submersion germ $\mu : (\mathbb{R}^2, 0) \longrightarrow \mathbb{R}$ such that $g = \widehat{\pi} \circ f$. Therefore we can apply the theory of big wave fronts. In [12] the following proposition has been shown.

Proposition 5.1 ([12]) *Let f_i $(i = 1, 2)$ be completely integrable first order ODEs with the integrals μ_i and the corresponding integral diagrams are (μ_i, g_i). Suppose that sets of Legendrian singular points of $\ell_{(\mu_i,f_i)}$ $(i = 1, 2)$ are nowhere dense. Then the following conditions are equivalent:*
(1) *f_1, f_2 are equivalent as ODEs.*
(2) *There exists a diffeomorphism germ $\Phi : (\mathbb{R}\times\mathbb{R}\times\mathbb{R}, 0) \longrightarrow (\mathbb{R}\times\mathbb{R}\times\mathbb{R}, 0)$ of the form $\Phi(t, x, y) = (\phi_1(t), \phi_2(x, y), \phi_3(x, y))$ such that $\hat{\Phi}(\mathrm{Image}\, \ell_{(\mu_1,f_1)}) = \mathrm{Image}\, \ell_{(\mu_2,f_2)}$.*
(3) *There exist diffeomorphism germs $\phi : (\mathbb{R}, 0) \longrightarrow (\mathbb{R}, 0)$, $\Phi : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^2, 0)$ and $\Psi : (\mathbb{R}\times\mathbb{R}, 0) \longrightarrow (\mathbb{R}\times\mathbb{R}, 0)$ such that $\phi \circ \mu_1 = \mu_2 \circ \Phi$ and $\Psi \circ g_1 = g_2 \circ \Phi$.*

We say that two integral diagrams (μ_1, g_1) and (μ_2, g_2) are *equivalent* as integral diagrams if the condition (3) of the above theorem holds. By Remark 3.7, the classification by the above equivalence is almost impossible. We also say that integral diagrams (μ_1, g_1) and (μ_2, g_2) are *strictly equivalent* if the condition (3) of the above theorem holds for $\phi = 1_{\mathbb{R}}$. The strict equivalence corresponds to the $S.P$-Legendrian equivalence among the big Legendrian submanifold germs $\ell_{(\mu,f)}$. Instead of the above equivalence relation, it was classified by $S.P$-Legendrian equivalence in [12]. The technique used there was very hard. In [16], it was classified by the $S.P^+$-Legendrian equivalence. If we have a classification of $\ell_{(\mu,f)}$ under the $S.P^+$-Legendrian equivalence, we can automatically obtain the classification of integral diagrams by the strict equivalence.

Theorem 5.2 ([12, 16]) *For a "generic"first order ODE $f : (\mathbb{R}^2, 0) \longrightarrow J^1(\mathbb{R},\mathbb{R})$ with a complete integral $\mu : (\mathbb{R}^2, 0) \longrightarrow \mathbb{R}$, the corresponding integral diagram (μ, g) is strictly equivalent to one of the germs in the following list:*

(1) $\mu = u_2$, $g = (u_1, u_2)$,
(2) $\mu = \dfrac{2}{3}u_1^3 + u_2$, $g = (u_1^2, u_2)$,
(3) $\mu = u_2 - \dfrac{1}{2}u_1$, $g = (u_1, u_2^2)$,
(4) $\mu = \dfrac{3}{4}u_1^4 + \dfrac{1}{2}u_1^2 u_2 + u_2 + \alpha \circ g$, $g = (u_1^3 + u_2 u_1, u_2)$,
(5) $\mu = u_2 + \alpha \circ g$, $g = (u_1, u_2^3 + u_1 u_2)$,
(6) $\mu = -3u_2^2 + 4u_1 u_2 + u_1 + \alpha \circ g$, $g = (u_1, u_2^3 + u_1 u_2^2)$.

Here, $\alpha(v_1, v_2)$ are C^∞-function germs, which are called functional modulus.

Remark 5.3 The results has been generalized into the case for completely integrable holonomic systems of first order partial differential equations [16, 17].

In the list of the above theorem, the normal forms (3), (5) are called *Clairaut type*. The complete solutions for those equations are non-singular and the singular solutions are the envelopes of the graph of complete solutions. We say that a complete integrable first order ODE $f : (\mathbb{R}^2, 0) \longrightarrow J^1(\mathbb{R}, \mathbb{R})$ with an integral $\mu : (\mathbb{R}^2, 0) \longrightarrow \mathbb{R}$ is *Clairaut type* if $\widehat{\pi} \circ f|_{\mu^{-1}(t)}$ is non-singular for any $t \in \mathbb{R}$. Then $\overline{\pi} \circ \ell_{(\mu,f)}$ is also non-singular. In this case the discriminant of the family $W_t(\ell_{(\mu,f)}(\mathbb{R}^2))$ is equal to the envelope of the family of momentary fronts $\Delta_{\ell_{(\mu,f)}(\mathbb{R}^2)}$. Here, the momentary front is a special solution of the complete solution $\{\widehat{\pi} \circ f(\mu^{-1}(t))\}_{t\in\mathbb{R}}$. This means that $\ell_{(\mu,f)}(\mathbb{R}^2) \cap J^1_{GA}(\mathbb{R} \times \mathbb{R}, \mathbb{R}) = \emptyset$.

On the other hand, the normal forms (2), (4) are called *regular type*. In this case $f^*\theta \neq 0$. In those cases, $\ell_{(\mu,f)}(\mathbb{R}^2) \subset J^1_{GA}(\mathbb{R} \times \mathbb{R}, \mathbb{R})$. Therefore, $\ell_{(\mu,f)}(\mathbb{R}^2)$ is a graph-like Legendrian unfolding, so that the discriminate of the family $W_t(\ell_{(\mu,f)}(\mathbb{R}^2))$ is $C_{\ell_{(\mu,f)}(\mathbb{R}^2)} \cup M_{\ell_{(\mu,f)}(\mathbb{R}^2)}$. Finally the normal form (6) is called a *mixed hold type*. In this case, $\ell_{(\mu,f)}(\mathbb{R}^2) \subset J^1(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ but $\ell_{(\mu,f)}(\mathbb{R}^2) \not\subset J^1_{GA}(\mathbb{R} \times \mathbb{R}, \mathbb{R})$. Actually, $\ell_{(\mu,f)}(0) \in \overline{J^1_{GA}(\mathbb{R} \times \mathbb{R}, \mathbb{R})}$, where $\overline{X}$ is the closure of X. The pictures of the families of momentary fronts of (4), (5), (6) are drawn in Figures 5, 6 and 7. We can observe that the discriminants of the families $W_t(\ell_{(\mu,f)}(\mathbb{R}^2))$ are $C_{\ell_{(\mu,f)}(\mathbb{R}^2)} \cup M_{\ell_{(\mu,f)}(\mathbb{R}^2)}$ for (4), $\Delta_{\ell_{(\mu,f)}(\mathbb{R}^2)}$ for (5) and $C_{\ell_{(\mu,f)}(\mathbb{R}^2)} \cup \Delta_{\ell_{(\mu,f)}(\mathbb{R}^2)}$ for (6), respectively. Moreover, the $C_{\ell_{(\mu,f)}(\mathbb{R}^2)}$ of the germ (4) and $\Delta_{\ell_{(\mu,f)}(\mathbb{R}^2)}$ of the germ (5) are semi-cubical parabolas. Therefore, these are diffeomorphic but those discriminants are not $S.P^+$-diffeomorphic.

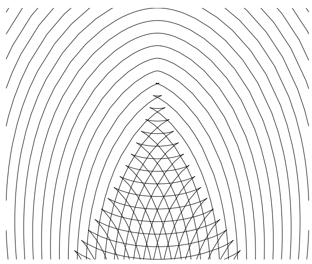


Fig.5: (4) Regular cusp

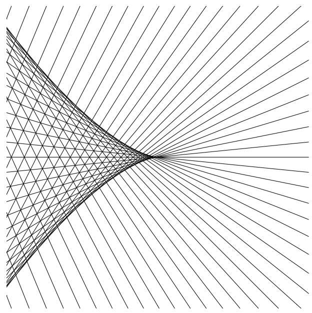


Fig.6: (5) Clairaut cusp

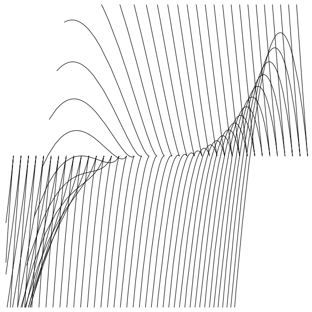


Fig.7: (6) Mixed fold

5.2 Quasi-linear first order partial differential equations

We consider a time-dependent quasi-linear first order partial differential equation

$$\frac{\partial y}{\partial t} + \sum_{i=1}^{n} a_i(x, y, t)\frac{\partial y}{\partial x_i} - b(x, y, t) = 0,$$

where $a_i(x, y, t)$ and $b(x, y, t)$ are C^∞-function of $(x, y, t) = (x_1, \ldots, x_n, y, t)$. In order to clarify the situation that there appeared a blow-up of the derivatives of solutions, we have constructed a geometric framework of the equation in [18]. A *time-dependent quasi-linear first order partial differential equation* is defined to be a hypersurface in $\overline{\pi} : PT^*((\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}) \longrightarrow (\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}$:

$$E(1, a_1, \ldots, a_n, b) = \{(x, y, t), [\xi : \eta : \sigma]) \mid \sigma + \sum_{i=1}^{n} a_i(x, y, t)\xi_i + b(x, y, t)\eta = 0 \ \}$$

A *geometric solution* of $E(1, a_1, \dots, a_n, b)$ is a Legendrian submanifold $\mathscr{L}$ of $PT^*((\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R})$ lying in $E(1, a_1, \dots, a_n, b)$ such that $\overline{\pi}|_{\mathscr{L}}$ is an embedding. Let S be a smooth hypersurface in $(\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}$. Then we have a unique Legendrian submanifold $\widehat{S}$ in $PT^*((\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R})$ such that $\overline{\pi}(\widehat{S}) = S$. It follows that if $\mathscr{L}$ is a geometric solution of $E(1, a_1, \dots, a_n, b)$, then $\mathscr{L} = \widehat{\overline{\pi}(\mathscr{L})}$. For any $(x_0, y_0, t_0) \in S$, there exists a smooth submersion germ $f : ((\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}, (x_0, y_0, t_0)) \longrightarrow (\mathbb{R}, 0)$ such that $(f^{-1}(0), (x_0, y_0, t_0)) = (S, (x_0, y_0, t_0))$ as set germs. A vector $\tau\partial/\partial t + \sum_{i=1}^n \mu_i \partial/\partial x_i + \lambda\partial/\partial y$ is tangent to S at $(x, y, t) \in (S, (x_0, y_0, t_0))$ if and only if $\tau\partial f/\partial t + \sum_{i=1}^n \mu_i \partial f/\partial x_i + \lambda\partial f/\partial y = 0$ at (x, y, t). Then we have the following representation of $\widehat{S}$:

$$(\widehat{S}, ((x_0, y_0, t_0), [\sigma_0 : \xi_0 : \eta_0])) = \left\{ \left((x, y, t), \left[\frac{\partial f}{\partial x} : \frac{\partial f}{\partial y} : \frac{\partial f}{\partial t} \right] \right) \middle| (x, y, t) \in (S, (x_0, y_0, t_0)) \right\}.$$

Under this representation, $\widehat{S} \subset E(1, a_1, \dots, a_n, b)$ if and only if

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{n} a_i(x, y, t) \frac{\partial f}{\partial x_i} + b(x, y, t) \frac{\partial f}{\partial y} = 0.$$

Here, the *characteristic vector field* of $E(1, a_1, \dots, a_n, b)$ is defined to be

$$X(1, a_1, \dots, a_n, b) = \frac{\partial}{\partial t} + \sum_{i=1}^{n} a_i(x, y, t) \frac{\partial}{\partial x_i} + b(x, y, t) \frac{\partial}{\partial y}.$$

In [18] a characterization theorem of geometric solutions has been proved.

Theorem 5.4 ([18]) *Let S be a smooth hypersurface in $(\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}$. Then $\widehat{S}$ is a geometric solution of $E(1, a_1, \dots, a_n, b)$ if and only if the characteristic vector field $X(1, a_1, \dots, a_n, b)$ is tangent to S.*

Remark 5.5 We consider Cauchy problem here:

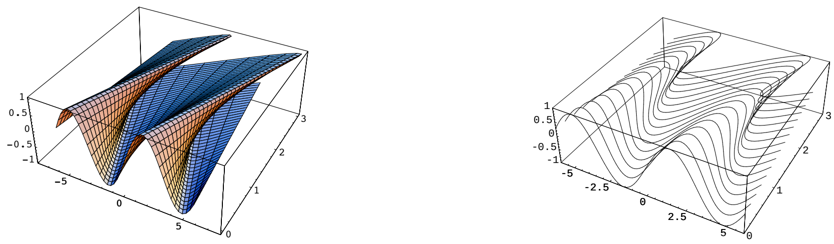
$$\frac{\partial y}{\partial t} + \sum_{i=1}^{n} a_i(x, y, t) \frac{\partial y}{\partial x_i} - b(x, y, t) = 0,$$
$$y(0, x_1, \dots, x_n) = \phi(x_1, \dots, x_n),$$

where ϕ is a C^∞-function. By Theorem 5.4, applying the classical method of characteristic, we can solve the above Cauchy problem. Although y is initially smooth, there is, in general, a critical time beyond which characteristics cross. After the characteristics cross, the geometric solution becomes multi-valued. Since the characteristic vector field $X(1, a_1, \dots, a_n, b)$ is a vector field on the space $(\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}$, the graph of the geometric solution $\overline{\pi}(\mathscr{L}) \subset (\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}$ is a smooth hypersurface. In general, however, $\widehat{\pi}_2|_{\overline{\pi}(\mathscr{L})}$ is a finite-to-one mapping, where $\widehat{\pi}_2 : (\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R} \longrightarrow \mathbb{R}^n \times \mathbb{R}$ is $\widehat{\pi}_2(x, y, t) = (x, t)$.

The geometric solution $\mathscr{L}$ is a big Legendrian submanifold and it is Legendrian nonsingular. Therefore, the discriminant of the family of the momentary fronts $W_t(\mathscr{L})$ is $\Delta_{\mathscr{L}}$. We consider the following example:

$$\frac{\partial y}{\partial t} + 2y \frac{\partial y}{\partial x} = 0,$$
$$y(0, x) = \sin x,$$

This equation is called *Burger's equation* and can be solved exactly by the characteristic method. We can draw the picture of the graph of the geometric solution and the family of $\pi_2^{-1}(t) \cap W(\mathscr{L})$ in Fig.8. We can observe that the graph is a smooth surface in $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$



The graph of the geometric solution of Burger's equation

The family of $\pi_2^{-1}(t) \cap W(\mathscr{L})$

Fig.8.

but it is multi-valued. Moreover, each $\pi_2^{-1}(t) \cap W(\mathscr{L})$ is non-singular but $\widehat{\pi}_2|_{\pi_2^{-1}(t)\cap W(\mathscr{L})}$ has singularities. Thus, $W(\mathscr{L})$ is a big wave front but not a graph-like wave front.

5.3 Parallels and Caustics of hypersurfaces in Euclidean space

In this subsection we consider the focal set (i.e., evolute) of hypersurface as the caustics and the parallels of hypersurface as the graph-like wave fronts by using the distance squared functions (cf. [19, 32]).

Let $\boldsymbol{X} : U \longrightarrow \mathbb{R}^n$ be an embedding, where U is an open subset in $\mathbb{R}^{n-1}$. We denote that $M = \boldsymbol{X}(U)$ and identify M and U through the embedding $\boldsymbol{X}$. The *Gauss map* is defined by the unit normal vector of M, namely, the Gauss map $\mathbb{G} : U \longrightarrow S^{n-1}$ is given by $\mathbb{G}(u) = \boldsymbol{n}(u)$, where $\boldsymbol{n}(u)$ is the unit normal vector of M at $\boldsymbol{X}(u)$. For a hypersurface $\boldsymbol{X} : U \longrightarrow \mathbb{R}^n$, we define the *focal set* (or, *evolute*) of $\boldsymbol{X}(U) = M$ by

$$\mathrm{F}_M = \left\{ \boldsymbol{X}(u) + \frac{1}{\kappa(u)}\boldsymbol{n}(u) \mid \kappa(u) \text{ is a principal curvature at } p = \boldsymbol{X}(u), u \in U \right\}$$

and the set of *unfolded parallels* of $\boldsymbol{X}(U) = M$ by

$$P_M = \{(\boldsymbol{X}(u) + r\boldsymbol{n}(u), r) \mid r \in \mathbb{R} \setminus \{0\}, u \in U\}.$$

We also define the smooth mapping $\mathrm{F}_\kappa : U \longrightarrow \mathbb{R}^n$ and $P_r : U \longrightarrow \mathbb{R}^n$ by

$$\mathrm{F}_\kappa(u) = \boldsymbol{X}(u) + \frac{1}{\kappa(u)}\boldsymbol{n}(u), \;\; P_r(u) = \boldsymbol{X}(u) + r\boldsymbol{n}(u),$$

where we fix a principal curvature $\kappa(u)$ on U at u with $\kappa(u) \neq 0$ and a real number $r \neq 0$.

We now define families of functions in order to describe the focal set and the parallels of a hypersurface in $\mathbb{R}^n$. We define

$$D : U \times (\mathbb{R}^n \setminus M) \longrightarrow \mathbb{R}$$

by $D(u, \boldsymbol{v}) = \|\boldsymbol{X}(u) - \boldsymbol{v}\|^2$ and

$$\overline{D} : U \times (\mathbb{R}^n \setminus M) \times \mathbb{R}_+ \longrightarrow \mathbb{R}$$

by $\overline{D}(u, \boldsymbol{v}, t) = \|\boldsymbol{X}(u) - \boldsymbol{v}\|^2 - t$, where we denote that $\mathbb{R}_+$ is the set of positive real numbers. We call D a *distance squared function* and $\overline{D}$ an *extended distance squared function* on $M = \boldsymbol{X}(U)$. Denote that the function $d_{\boldsymbol{v}}$ and $\overline{d}_{\boldsymbol{v}}$ by $d_{\boldsymbol{v}}(u) = D(u, \boldsymbol{v})$ and $\overline{d}_{\boldsymbol{v}}(u, t) = \overline{D}(u, \boldsymbol{v}, t)$ respectively.

The following proposition follows from straightforward calculations (cf., [19]):

Proposition 5.6 *Let $\boldsymbol{X} : U \longrightarrow \mathbb{R}^n$ be a hypersurface. Then*

(1) $(\partial d_{\boldsymbol{v}}/\partial u_i)(u) = 0$ $(i = 1, \ldots, n-1)$ *if and only if there exists a real number $r \in \mathbb{R} \setminus \{0\}$ such that $\boldsymbol{v} = \boldsymbol{X}(u) + r\boldsymbol{n}(u)$.*

(2) $(\partial d_{\boldsymbol{v}}/\partial u_i)(u) = 0$ $(i = 1, \ldots, n-1)$ *and* $\det(\mathcal{H}(d_v)(u)) = 0$ *if and only if $\boldsymbol{v} = \boldsymbol{X}(u) + (1/\kappa(u))\boldsymbol{n}(u)$.*

(3) $\overline{d}_{\boldsymbol{v}}(u, t) = (\partial \overline{d}_{\boldsymbol{v}}/\partial u_i)(u, t) = 0$ $(i = 1, \ldots, n-1)$ *if and only if $\boldsymbol{v} = \boldsymbol{X}(u) \pm \sqrt{t}\boldsymbol{n}(u)$.*

Here $\mathcal{H}(d_{\boldsymbol{v}})(u)$ is the hessian matrix of the function $d_{\boldsymbol{v}}$ at u.

We can detect the catastrophe set and bifurcation set of distance squared function D and the discriminant set of the extended distance squared function $\overline{D}$ by Proposition 5.6.

$$C(D) = \{(u, \boldsymbol{v}) \in U \times (\mathbb{R}^n \setminus M) \mid \boldsymbol{v} = \boldsymbol{X}(u) + r\boldsymbol{n}(u),\ r \in \mathbb{R} \setminus \{0\}\}$$

and

$$\Sigma_*(\overline{D}) = \left\{(u, \boldsymbol{v}, t) \in U \times (\mathbb{R}^n \setminus M) \times \mathbb{R}_+ \mid \boldsymbol{v} = \boldsymbol{X}(u) \pm \sqrt{t}\boldsymbol{n}(u),\ u \in U\right\}.$$

We can naturally interpret the focal set of a hypersurface as a caustic and parallels of a hypersurface as graph-like wave fronts (big wave front).

Proposition 5.7 *For a hypersurface $\boldsymbol{X} : U \longrightarrow \mathbb{R}^n$, the distance squared function $D : U \times (\mathbb{R}^n \setminus M) \longrightarrow \mathbb{R}$ is a Morse family of functions and the extended distance squared function $\overline{D} : U \times (\mathbb{R}^n \setminus M) \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ is a non-degenerate graph-like Morse family of hypersurfaces.*

Proof. By Proposition 4.5, it is enough to show that D is a Morse family of hypersurfaces. For any $\boldsymbol{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n \setminus M$, we have $D(u, \boldsymbol{v}) = \sum_{i=1}^n (x_i(u) - v_i)^2$, where $\boldsymbol{X}(u) = (x_1(u), \ldots, x_n(u))$. We shall prove that the mapping

$$\Delta^* D = \left(D, \frac{\partial D}{\partial u_1}, \ldots, \frac{\partial D}{\partial u_{n-1}}\right)$$

is a non-singular at any point. The Jacobian matrix of $\Delta^* D$ is given by

$$\begin{pmatrix} A_1(u) & \cdots & A_{n-1}(u) & -2(x_1(u) - v_1) & \cdots & -2(x_{n-1} - v_{n-1}) \\ A_{11}(u) & \cdots & A_{1(n-1)}(u) & -2x_{1u_1}(u) & \cdots & -2x_{nu_1}(u) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{(n-1)1}(u) & \cdots & A_{(n-1)(n-1)} & -2x_{1u_{n-1}}(u) & \cdots & -2x_{nu_{n-1}}(u) \end{pmatrix},$$

where $A_i(u) = \langle 2\boldsymbol{X}_{u_i}(u), \boldsymbol{X}(u)-\boldsymbol{v}\rangle$, $A_{ij}(u) = 2(\langle \boldsymbol{X}_{u_iu_j}(u), \boldsymbol{X}(u)-\boldsymbol{v}\rangle + \langle \boldsymbol{X}_{u_i}(u), \boldsymbol{X}_{u_j}(u)\rangle)$ and $\langle , \rangle$ is the inner product of $\mathbb{R}^n$. Suppose that $(u, \boldsymbol{v}, t_0) \in \Sigma_*(\overline{D})$. Then we have $\boldsymbol{v} = \boldsymbol{X}(u) \pm \sqrt{t_0}\boldsymbol{n}(u)$. Therefore, we have

$$J_{\Delta^* D}(u, \boldsymbol{v}, t_0) = \begin{pmatrix} 0 & \mp 2\sqrt{t_0}\boldsymbol{n}(u) \\ A_{ij}(u) & -2\boldsymbol{X}_{u_i}(u) \end{pmatrix}.$$

Since $\boldsymbol{n}(u), \boldsymbol{X}_{u_1}(u), \dots, \boldsymbol{X}_{u_{n-1}}(u)$ are linearly independent, the rank of $J_{\Delta^* D}(u, \boldsymbol{v}, t_0)$ is n. This means that D is a Morse family of hypersurfaces. □

By the method for constructing the Lagrangian submanifold germ from Morse family of functions (cf. §2), we can define a Lagrangian submanifold germ whose generating family is the distance squared function D of $M = \boldsymbol{X}(U)$ as follows: For a hypersurface $\boldsymbol{X} : U \longrightarrow \mathbb{R}^n$ where $\boldsymbol{X}(u) = (x_1(u), \dots, x_n(u))$, we define

$$L(D) : C(D) \longrightarrow T^*\mathbb{R}^n$$

by

$$L(D)(u, \boldsymbol{v}) = (\boldsymbol{v}, -2(x_1(u) - v_1), \dots, -2(x_n(u) - v_n)),$$

where $\boldsymbol{v} = (v_1, \dots, v_n)$.

On the other hand, by the method for constructing the graph-like Legendrian unfolding from a graph-like Morse family of hypersrufaces (cf. §4), we can define a graph-like Legendrian unfolding whose generating family is the extended distance squared function $\overline{D}$ of $M = \boldsymbol{X}(U)$. For a hypersurface $\boldsymbol{X} : U \longrightarrow \mathbb{R}^n$ where $\boldsymbol{X}(u) = (x_1(u), \dots, x_n(u))$, we define

$$\mathfrak{L}_D : C(D) \longrightarrow J^1_{GA}(\mathbb{R}^n, \mathbb{R})$$

by

$$\mathfrak{L}_D(u, \boldsymbol{v}) = (\boldsymbol{v}, \|\boldsymbol{X}(u) - \boldsymbol{v}\|^2, -2(x_1(u) - v_1), \dots, -2(x_n(u) - v_n)),$$

where $\boldsymbol{v} = (v_1, \dots, v_n)$.

Corollary 5.8 *Under the above notations, $L(D)(C(D))$ is a Lagrangian submanifold such that the distance squared function D is a generating family of $L(D)(C(D))$ and $\mathfrak{L}_D(C(D))$ is a non-degenerate graph-like Legendrian unfolding such that the extended distance squared function $\overline{D}$ is a graph-like generating family of $\mathfrak{L}_D(C(D))$.*

By Proposition 5.6, the caustic $C_{L(D)(C(D))}$ of $L(D)(C(D))$ is the focal set F_M and the graph-like wave front $W(\mathfrak{L}_D(C(D)))$ is the set of unfolded parallels P_M.

We now briefly describe the theory of contact with foliations. Here we consider the relationship between the contact of submanifolds with foliations and the $\mathcal{R}^+$-class of functions. Let X_i $(i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$, $g_i : (X_i, \bar{x}_i) \longrightarrow (\mathbb{R}^n, \bar{y}_i)$ be immersion germs and $f_i : (\mathbb{R}^n, \bar{y}_i) \longrightarrow (\mathbb{R}, 0)$ be submersion germs. For a submersion germ $f : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}, 0)$, we denote that $\mathcal{F}_f$ be the regular foliation defined by f; i.e., $\mathcal{F}_f = \{f^{-1}(c) | c \in (\mathbb{R}, 0)\}$. We say that *the contact of X_1 with the regular foliation $\mathcal{F}_{f_1}$* at $\bar{y}_1$ is of the *same type* as *the contact of X_2 with the regular foliation $\mathcal{F}_{f_2}$* at $\bar{y}_2$ if there is a diffeomorphism germ $\Phi : (\mathbb{R}^n, \bar{y}_1) \longrightarrow (\mathbb{R}^n, \bar{y}_2)$ such that $\Phi(X_1) = X_2$ and $\Phi(Y_1(c)) = Y_2(c)$, where $Y_i(c) = f_i^{-1}(c)$ for each $c \in (\mathbb{R}, 0)$. In this case we write

$K(X_1, \mathcal{F}_{f_1}; \bar{y}_1) = K(X_2, \mathcal{F}_{f_2}; \bar{y}_2)$. It is also clear that in the definition $\mathbb{R}^n$ could be replaced by any manifold. We apply the method of Goryunov[10] to the case for $\mathcal{R}^+$-equivalences among function germs, so that we have the following:

Proposition 5.9 ([10, Appendix]) *Let X_i $(i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2 = n-1$ (i.e. hypersurface), $g_i : (X_i, \bar{x}_i) \longrightarrow (\mathbb{R}^n, \bar{y}_i)$ be immersion germs and $f_i : (\mathbb{R}^n, \bar{y}_i) \longrightarrow (\mathbb{R}, 0)$ be submersion germs. Then $K(X_1, \mathcal{F}_{f_1}; \bar{y}_1) = K(X_2, \mathcal{F}_{f_2}; \bar{y}_2)$ if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $\mathcal{R}^+$-equivalent.*

On the other hand, we define a function $\mathcal{D} : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ by $\mathcal{D}(\boldsymbol{x}, \boldsymbol{v}) = \|\boldsymbol{x} - \boldsymbol{v}\|^2$. For any $\boldsymbol{v} \in \mathbb{R}^n \setminus M$, we denote that $\mathfrak{d}_{\boldsymbol{v}}(\boldsymbol{x}) = \mathcal{D}(\boldsymbol{x}, \boldsymbol{v})$ and we have a hypersphere $\mathfrak{d}_{\boldsymbol{v}}{}^{-1}(c) = S^{n-1}(\boldsymbol{v}, \sqrt{c})$ for any $c > 0$. It is easy to show that $\mathfrak{d}_{\boldsymbol{v}}$ is a submersion. For any $u \in U$, we consider a point $\boldsymbol{v}^{\pm} = \boldsymbol{X}(u) \pm \sqrt{c}\boldsymbol{n}(u) \in \mathbb{R}^n \setminus M$, then we have

$$\mathfrak{d}_{\boldsymbol{v}^{\pm}} \circ \boldsymbol{X}(u) = \mathcal{D} \circ (\boldsymbol{X} \times id_{\mathbb{R}^n})(u, \boldsymbol{v}^{\pm}) = c,$$

and

$$\frac{\partial \mathfrak{d}_{\boldsymbol{v}^{\pm}} \circ \boldsymbol{X}}{\partial u_i}(u) = \frac{\partial D}{\partial u_i}(u, \boldsymbol{v}^{\pm}) = 0.$$

for $i = 1, \dots, n-1$. This means that the hypersphere $\mathfrak{d}_{\boldsymbol{v}^{\pm}}^{-1}(r) = S^{n-1}(\boldsymbol{v}^{\pm}, \sqrt{c})$ is tangent to $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u)$. In this case, we call one of $S^{n-1}(\boldsymbol{v}^{\pm}, \sqrt{c})$ a *tangent hypersphere* at $p = \boldsymbol{X}(u)$ with the center $\boldsymbol{v}^{\pm}$. However, there are infinitely many tangent hyperspheres at a general point $p = \boldsymbol{X}(u)$ depending on the real number c. If $\boldsymbol{v}$ is a point of the focal set (i.e., $\boldsymbol{v} = F_{\kappa}(u)$ for some κ), the tangent hypersphere with the center $\boldsymbol{v}$ is called the *osculating hypersphere* (or, *curvature hypersphere*) at $p = \boldsymbol{X}(u)$ which is uniquely determined. For $\boldsymbol{v}^{\pm} = \boldsymbol{X}(u) \pm \sqrt{c}\boldsymbol{n}(u)$, we also have regular foliations

$$\mathcal{F}_{\mathfrak{d}_{\boldsymbol{v}^{\pm}}} = \left\{ S^{n-1}(\boldsymbol{v}^{\pm}, \sqrt{t}) \;\middle|\; t \in (\mathbb{R}, c) \right\}$$

whose leaves are hyperspheres with the center $\boldsymbol{v}^{\pm}$ such that the case $t = c$ corresponding to the tangent hyperspheres with radius $|c|$. Moreover, if $\boldsymbol{v} = F_{\kappa}(u)$, then $S^{n-1}(\boldsymbol{v}, 1/\kappa(u))$ is the osculating hypersphere. In this case $(\boldsymbol{X}^{-1}(\mathcal{F}_{\mathfrak{d}_{\boldsymbol{v}}}), u)$ is a singular foliation germ at u which is called a *osculating hyperspherical foliation* of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u)$ (or, u). We denote it by $\mathcal{OF}(M, u)$. Moreover, if $\boldsymbol{v} \in M_{\mathfrak{L}_D(C(D))}$, then there exists $r_0 \in \mathbb{R} \setminus \{0\}$ such that $(\boldsymbol{v}, r_0)$ is a self-intersection point of P_M, so that there exist different $u, v \in U$ such that $\boldsymbol{v} = \boldsymbol{X}(u) + r_0\boldsymbol{n}(u) = \boldsymbol{X}(v) + r_0\boldsymbol{n}(v)$. Therefore, the hypersphere $S^{n-1}(\boldsymbol{v}, |r_0|)$ is tangent to $M = \boldsymbol{X}(U)$ at both the points $p = \boldsymbol{X}(u)$ and $q = \boldsymbol{X}(v)$. Then we have an interpretation of the geometric meanings of the Maxwell stratified set in this case:

$$M_{\mathfrak{L}_D(C(D))} = \{\boldsymbol{v} \mid \exists r_0 \in \mathbb{R}\backslash\{0\}, S^{n-1}(\boldsymbol{v}, |r_0|) \text{ is tangent to } M \text{ at least two different points}\}.$$

Therefore, we call the Maxwell stratified set $M_{\mathfrak{L}_D(C(D))}$ the *set of the centers of multiple tangent spheres* of M.

We consider the contact of hypersurfaces with families of hyperspheres. Let $\boldsymbol{X}_i : (U, \bar{u}_i) \longrightarrow (\mathbb{R}^n, p_i)$ $(i = 1, 2)$ be hypersurface germs. We consider distance squared functions $D_i : (U \times \mathbb{R}^n, (\bar{u}_i, \boldsymbol{v}_i)) \longrightarrow \mathbb{R}$ of $M_i = \boldsymbol{X}_i(U)$, where $\boldsymbol{v}_i = \mathrm{Ev}_{\kappa_i}(\bar{u}_i)$. We denote that $d_{i,\boldsymbol{v}_i}(u) = D_i(u, \boldsymbol{v}_i)$, then we have $d_{i,\boldsymbol{v}_i}(u) = \mathfrak{d}_{\boldsymbol{v}_i} \circ \boldsymbol{X}_i(u)$. Then we have the following theorem:

Theorem 5.10 *Let* $\boldsymbol{X}_i : (U, \bar{u}_i) \longrightarrow \mathbb{R}^n, p_i)$ $(i = 1, 2)$ *be hypersurface germs such that the corresponding graph-like Legendrian unfolding germs* $\mathfrak{L}_{D_i}(C(D_i))$ *are* $S.P^+$*-Legendrian stable (i.e., the corresponding Lagrangian submanifold germs* $L(D_i)(C(D_i))$ *are Lagrangian stable), where* $\boldsymbol{v}_i = \mathrm{Ev}_{\kappa_i}(\bar{u}_i)$ *are centers of the osculating hyperspheres of* $M_i = \boldsymbol{X}_i(U)$ *respectively. Then the following conditions are equivalent:*
(1) $\mathfrak{L}_{D_1}(C(D_i))$ *and* $\mathfrak{L}_{D_i}(C(D_2))$ *are* $S.P^+$*-Legendrian equivalent,*
(2) $\overline{D}_1$ *and* $\overline{D_2}$ *are s-*$S.P^+$*-*$\mathcal{K}$*-equivalent,*
(3) $\bar{d}_{1,\boldsymbol{v}_1}$ *and* $\bar{d}_{1,\boldsymbol{v}_2}$ *are* $S.P$*-*$\mathcal{K}$*-equivalent,*
(4) $d_{1,\boldsymbol{v}_1}$ *and* $d_{2,\boldsymbol{v}_2}$ *are* $\mathcal{R}$*-equivalent,*
(5) $K(M_1, \mathcal{F}_{\mathfrak{d}_{\boldsymbol{v}_1}}; p_1) = K(M_2, \mathcal{F}_{\mathfrak{d}_{\boldsymbol{v}_2}}; p_2)$*,*
(6) D_1 *and* D_2 *are* P*-*$\mathcal{R}^+$*-equivalent,*
(7) $L(D_1)(C(D_1))$ *and* $L(D_2)(C(D_2))$ *are Lagrangian equivalent,*
(8) P_{M_1} *and* P_{M_2} *are* $S.P^+$*-diffeomorphic.*

Proof. By Theorem 5.9, the conditions (4) and (5) are equivalent. By the assertion (3) of Proposition 5.6, we have $W(\mathfrak{L}_{D_i}(C(D_i)) = P_{M_i}$. Thus, the other conditions are equivalent by Theorem 4.4. □

We remark that if $L(D_1)$ and $L(D_2)$ are Lagrangian equivalent, then the corresponding caustics are diffeomorphic. Since the caustic of $L(D)$ is the focal set of a hypersurface $M = \boldsymbol{X}(U)$, the above theorem gives a symplectic interpretation for the contact of hypersurfaces with family of hyperspheres. Moreover, the $S.P^+$-diffeomorphism between the graph-like wave front sets sends the Maxwell stratified sets to each other. Therefore, we have the following corollary.

Corollary 5.11 *Under the same assumptions as those of the above theorem for hypersurface germs* $\boldsymbol{X}_i : (U, \bar{u}_i) \longrightarrow (\mathbb{R}^n, p_i)$ $(i = 1, 2)$*, we have the following: If one of the conditions of the above theorem is satisfied, then*

(1) *The focal sets* F_{M_1} *and* F_{M_2} *are diffeomorphic as set germs.*

(2) *The osculating hyperspherical foliation germs* $\mathcal{OF}(M_1, \bar{u}_1)$*,* $\mathcal{OF}(M_2, \bar{u}_2)$ *are diffeomorphic.*

(3) *The sets of the centers of multiple tangent spheres of* M_1 *and* M_2 *are diffeomorphic as set germs.*

5.4 Caustics of world sheets

Recently the author has discovered application of the theory of graph-like Legendrian unfoldings to the caustics of world sheets in Lorentz space forms. In the theory of relativity, we do not have the notion of time constant, so that everything are moving depends on the time. Therefore, we have to consider world sheets instead of spacelike submanifolds. Let $\mathbb{L}_1^{n+1}$ be an $n+1$-dimensional Lorentz space form (i.e., Lorentz-Minkowski space, de Sitter space or anti-de Sitter space). For basic concepts and properties of Lorentz space forms, see [31]. We say that a non-zero vector $\boldsymbol{x} \in \mathbb{L}_1^{n+1}$ is *spacelike, lightlike or timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ or $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$ respectively. Here, $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ is the induced pseudo-scalar product of $\mathbb{L}_1^{n+1}$. We only consider the local situation here. Let $\boldsymbol{X} : U \times I \longrightarrow \mathbb{L}_1^{n+1}$ be a timelike embedding of codimension $k-1$, where $U \subset \mathbb{R}^s$ $(s + k = n + 1)$ is an open subset and I an open interval. We write $W = \boldsymbol{X}(U \times I)$ and identify W and $U \times I$ through the embedding $\boldsymbol{X}$. Here, the embedding $\boldsymbol{X}$ is said to be *timelike* if the tangent space T_pW of W at $p = \boldsymbol{X}(u,t)$ is a timelike subspace (i.e., Lorentz subspace of $T_p\mathbb{L}^{n+1}_1$) for any point $p \in W$. We denote that $\mathcal{S}_t = \boldsymbol{X}(U \times \{t\})$ for each $t \in I$. We call $\mathscr{F}_S = \{\mathcal{S}_t \mid t \in I\}$ a *spacelike foliation* on W if $\mathcal{S}_t$ is a spacelike submanifold for any $t \in I$. Here, we say that $\mathcal{S}_t$ is *spacelike* if the tangent space $T_p\mathcal{S}_t$ consists only spacelike vectors (i.e., spacelike subspace) for any point $p \in \mathcal{S}_t$. We call $\mathcal{S}_t$ a *momentary space* of $\mathscr{F}_S = \{\mathcal{S}_t \mid t \in I\}$. We say that $W = \boldsymbol{X}(U \times I)$ (or, $\boldsymbol{X}$ itself) is a *world sheet* if W is time-orientable. It follows that there exists a unique timelike future directed unit normal vector field $\boldsymbol{n}^T(u,t)$ along $\mathcal{S}_t$ on W (cf., [31]). It means that $\boldsymbol{n}^T(u,t) \in T_pW$ and psuedo-orthogonal to $T_p\mathcal{S}_t$ for $p = \boldsymbol{X}(u,t)$. Since T_pW is a timelike subspace of $T_p\mathbb{L}^{n+1}_1$, the pseudo-normal space $N_p(W)$ of W is a $k-1$-dimensional spacelike subspace of $T_p\mathbb{L}^{n+1}_1$ (cf.,[31]). On the pseudo-normal space $N_p(W)$, we have a $(k-2)$-unit sphere

$$N_1(W)_p = \{\boldsymbol{\xi} \in N_p(W) \mid \langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle = 1 \}.$$

Therefore, we have a unit spherical normal bundle over W:

$$N_1(W) = \bigcup_{p \in W} N_1(W)_p.$$

For an each momentary space $\mathcal{S}_t$, we have a unit spherical normal bundle $N_1[\mathcal{S}_t] = N_1(W)|_{\mathcal{S}_t}$ over $\mathcal{S}_t$. Then we define a hypersurface $\mathbb{LH}_{\mathcal{S}_t} : N_1[\mathcal{S}_t] \times \mathbb{R} \longrightarrow \mathbb{L}^{n+1}_1$ by

$$\mathbb{LH}_{\mathcal{S}_t}(((u,t),\boldsymbol{\xi}),\mu) = \boldsymbol{X}(u,t) + \mu(\boldsymbol{n}^T(u,t) + \boldsymbol{\xi}),$$

where $p = \boldsymbol{X}(u,t)$, which is called the *lightlike hypersurface* in the Lorentz space form $\mathbb{L}^{n+1}_1$ along $\mathcal{S}_t$. The lightlike hypersurface of a spacelike submanifold in a Lorentz space form has been defined and investigated in [21, 22, 23]. The singular value set of the lightlike hypersurface is called a *focal set* of $\mathcal{S}_t$, which is the Lorentz version of the focal set (or the caustics) in the Riemannian space form. However, the situation is different from the Riemannian case. The lightlike hypersurface is a wave front in $\mathbb{L}^{n+1}_1$, so that the focal set is the set of Legendrian singular values. In the Riemanniann case, the focal set is the set of Lagrangian singular values. In the Lorentzian case, we consider world sheets instead of single spacelike submanifold. Since a world sheet is a one-parameter family of spacelike submanifolds, we can naturally apply the theory of wave front propagations. We define

$$\widetilde{\mathbb{LH}}(W) = \bigcup_{t \in I} \mathbb{LH}_{\mathcal{S}_t}(N_1[\mathcal{S}_t] \times \mathbb{R}) \times \{t\} \subset \mathbb{L}^{n+1}_1 \times I,$$

which is called a *unfolded lightlike hypersurface*. In [28] we show that the unfolded lightlike hypersurface is a graph-like wave front and each lightlike hypersurface is a momentary front for the case that $\mathbb{L}^{n+1}_1$ is the anti-de Sitter space. One of the motivations to investigate this case is given in the brane world scenario (cf., [4, 5]). It has been considered lightlike hypersurfaces and the caustics along world sheets for the simplest case in their papers. Since the unfolded lightlike hypersurface is a graph-like Legendrian unfolding, we can investigate not only the caustic but also the Maxwel stratified set as an application of the theory of Legendrian unfoldings. We can apply Theorem 4.1 to this case and get some geometric information on world sheets. We can also consider the *lightcone pedal* of world sheets and investigate the geometric properties as an application of the theory of graph-like unfoldings [20, 27].

5.5 Control theory

In [36, 37] Zakalyukin applied the $S.P^+$-Legendrian equivalence for the study of problems occur in the control theory. In [36] he has given the following simple example: Consider a plane $\mathbb{R}^2$. For each point $q = (q_1, q_2) \in \mathbb{R}^2$, We consider an admissible curve on the tangent plane $\mathbb{R}^2 = T_q\mathbb{R}^2$ defined by $p_1 = 1 + u, p_2 = u^2$ $(u \in \mathbb{R})$, where $(p_1, p_2) \in \mathbb{R}^2$ is the coordinates of $\mathbb{R}^2 = T_q\mathbb{R}^2$. So this admissible curve is independent of the base point $q \in \mathbb{R}^2$. The initial front is given by $W_0 = \{(q_1, f(q_1)) \mid q_1 \in \mathbb{R}\}$ for some function $f(q_1)$. According to the Pontryagin maximum principle, externals of the corresponding time optimal contrail problem are defined by the canonical system of equations with the Hamiltonian $H(p, q) = \max_u(p_1(1 + u) + p_2u^2)$. This system can be solved exactly and the corresponding family of fronts W_t are given parametrically in the form $W_t = \Phi_t(W_0)$:

$$\Phi_t(q_1, t) = \left(q_1 + t\left(1 + \frac{1}{2}\frac{df}{dq_1}\right), f(q_1) + \frac{t}{4}\left(\frac{df}{dq_1}\right)^2 \right).$$

Under the condition $f'(0) = 0$ and $f''(0) > 0$, he has shown that the picture of the discriminant set of the family $\{W_t\}_{t\in I}$ is the same as that of the discriminant set of the germ (6) of Theorem 5.2. He also apply the $S.P^+$-Legendrian equivalence to translation-invariant control problems in [37].

The author is not a specialist of the control theory, so that we cannot explain the detail of the results here. However, it seems that there might be a lot of applications of the theory of wave front propagations to this area. For the detailed arguments, see the original articles.

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