On the Nash modification of a germ of complex analytic singularity

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Outline

Introduction

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- By taking a representative of $X \subset \mathbb{C}^n$ we can construct $\mathcal{N}X$ as an analytic subvariety of $\mathbb{C}^n \times G(d,n)$.
- Objective: Identify the subvarieties $Z \subset \mathbb{C}^n \times G(d,n)$ that are the Nash modification of their image under the canonical projection to $\mathbb{C}^n$.

Hypersurface case

When X is a hypersurface

- The Grasmannian $G(n-1, n)$ is the dual projective space $\check{\mathbb{P}}^{n-1}$ and the set $\nu^{-1}(0)$ is described via projective duality by a finite family of subcones of the tangent cone which include its irreducible components. (Lê & Teissier, 1988).

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- The generalization of this result to germs of arbitrary codimension needs to replace the Nash modification $\mathcal{N}X$ by the conormal space $C(X)$. (Limits of tangent hyperplanes).

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- Key: Identify $\mathbb{C}^n \times \check{\mathbb{P}}^{n-1}$ with the projectivized cotangent bundle of $\mathbb{C}^n$ and endow it with the canonical contact structure.
- $Z \subset \mathbb{C}^n \times \check{\mathbb{P}}^{n-1}$ is the conormal space of its image if and only if it is a Legendrian subvariety. (Integral subvariety of dimension $n-1$)

Example

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- Its tangent cone $C_{E_6,0}$ defined by $\{z_3^2 + z_4^2 = 0\}$ is reduced, having the $z_1 z_2$ plane as its singular locus and we have that: $\mathbb{P}\check{C}_{E_6,0} = \{[0:0:c:d] \mid c^2 + d^2 = 0\} \subset \check{\mathbb{P}}^3$.

Example

- However, the arc $\gamma : (\mathbb{C},0) \to (E_6,0)$ defined by $\tau \to (-\tau^4, \tau^3, 0, 0)$ lifts to the conormal space $C(E_6)$ as the arc:
$$\tau \to (\gamma(\tau), [3\tau^8 : 4\tau^9 : 0 : 0])$$
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- But $[1 : 0 : 0 : 0]$ is not in the dual of the tangent cone, so it must be in the dual of an exceptional cone!!
- Fact: $\kappa^{-1}(0) = \{[a : 0 : c : d]\} \subset \check{\mathbb{P}}^3$, with the exceptional cones being the $z_1 z_2$ plane and the z_2 axis.

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- Problem: Not every $\check{\mathbb{P}}^{n-d-1} \subset \kappa^{-1}(0)$ corresponds to a $T \in \nu^{-1}(0)$ and we don't know how to identify them.

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- The dimension of $\nu^{-1}(0)$ is at most 1.
- But the dimension of the set of planes of $\mathbb{C}^5$ that contain the line ℓ is 3.
- There are too many!!!

The canonical contact structure

- For a point $(p, W) \in \mathbb{C}^n \times \check{\mathbb{P}}^{n-1}$ the tangent space $T_{(p,W)}(\mathbb{C}^n \times \check{\mathbb{P}}^{n-1}) = \mathbb{C}^n \times T_W\check{\mathbb{P}}^{n-1}$.

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- For example in the chart of $\mathbb{C}^n \times \check{\mathbb{P}}^{n-1}$ where $a_1 \neq 0$:

$$dz_1 + \frac{a_2}{a_1}dz_2 + \cdots + \frac{a_n}{a_1}dz_n$$

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- It is locally defined by the kernel of a system of analytic 1-forms of $\mathbb{C}^n \times G(d,n)$.

Integral Subvarieties

- The analytic subvariety $Z \subset \mathbb{C}^n \times G(d,n)$ is an **integral subvariety** of $(\mathbb{C}^n \times G(d,n), \mathcal{H})$ if for every smooth point $(p, W) \in Z$ we have $T_{(p,W)}Z \subset \mathcal{H}(p,W)$.

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Proposition

Let $\pi : \mathbb{C}^n \times G(d,n) \to \mathbb{C}^n$ be the projection onto $\mathbb{C}^n$. If $Z \subset \mathbb{C}^n \times G(d,n)$ is an integral subvariety of $(\mathbb{C}^n \times G(d,n), \mathcal{H})$ then $t := \dim \pi(Z) \leq d$ and $\dim Z \leq t + (d-t)(n-d)$.

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- That is $d-$dimensional linear subspaces W of $\mathbb{C}^n$ such that $W \supset T_p\pi(Z)$.
- Generalization of both the Nash modification and the conormal space of a germ of singularity $(X, 0) \subset (\mathbb{C}^n, 0)$ where we consider limiting d-dimensional linear tangent spaces for any d in $\{\dim X, \ldots, n-1\}$.

The d-conormal space $C_d(X)$

Definition

Let $(X,0) \subset (\mathbb{C}^n, 0)$ be a germ of analytic,reduced and irreducible analytic singularity of dimension k. For any $d \in \{k, k+1, \ldots, n-1\}$ define the $d-$*conormal* of X by

$$C_d(X) := \overline{\{(z,W) \in X^0 \times G(d,n) \,|\, T_z X^0 \subset W\}}$$

where X^0 denotes the smooth part of X, $G(d,n)$ is the Grassmann variety of $d-$dimensional linear subspaces of $\mathbb{C}^n$ and the bar denotes closure in $\mathbb{C}^n \times G(d,n)$. We will denote by $\kappa_d : C_d(X) \to X$ the restriction of the projection to the first coordinate.

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- For $d = n-1$ we get the conormal space

$$\kappa : C(X) \to X$$

Characterization

Theorem

Let $Z \subset \mathbb{C}^n \times G(d,n)$ be a reduced, analytic and irreducible subvariety and $X = \pi(Z)$ where $\pi : \mathbb{C}^n \times G(d,n) \to \mathbb{C}^n$ denotes the projection to $\mathbb{C}^n$. If the dimension of X is equal to t, then the following statements are equivalent:

i) *Z is the d-conormal space of $X \subset \mathbb{C}^n$.*

ii) *Z is an integral subvariety of $(\mathbb{C}^n \times G(d,n), \mathcal{H})$ of dimension $t + (d-t)(n-d)$*

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- We recover the characterization of conormal varieties as legendrian subvarieties of $\mathbb{C}^n \times \check{\mathbb{P}}^{n-1}$ with its canonical contact structure.

Corollary

Let Z be an integral subvariety of $(\mathbb{C}^n \times G(d,n), \mathcal{H})$ of dimension d. Then Z is the Nash modification of its image in $\mathbb{C}^n$ if and only if for every smooth point $(z,W) \in Z^0$ the tangent space $T_{(z,W)}Z$ is transverse to the subspace $T_W G(d,n)$ of $T_{(z,W)}(\mathbb{C}^n \times G(d,n))$.

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- The transversality condition is there to prevent a drop in dimension from Z to $\pi(Z)$.

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- Consider $(X, 0) \subset (\mathbb{C}^n, 0)$ germ of analytic, reduced and irreducible singularity of dimension d.
- With $(Y, 0) \subset (X, 0)$ singular locus such that $(Y, 0)$ is smooth.

Proposition

Let $\mathcal{I}$ denote the ideal of $O_{\mathcal{N}X}$ that defines the intersection $C_d(Y) \cap \mathcal{N}X$ and J the ideal defining $\nu^{-1}(Y)$.

1. *The couple $(X \setminus Y, Y)$ satisfies Whitney's condition a) at the origin if and only if at every point $(0,T) \in \nu^{-1}(0)$ $\sqrt{\mathcal{I}} = \sqrt{J}$ in $O_{\mathcal{N}X,(0,T)}$.*
2. *The couple $(X \setminus Y, Y)$ satisfies condition Whitney conditions a) and b) at the origin if and only if at every point $(0,T) \in \nu^{-1}(0)$ the ideals $\mathcal{I}$ and J have the same integral closure in $O_{\mathcal{N}X,(0,T)}$.*

Thank you for listening.