Computing the homology of disentanglement of a germ of corank 2.

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1 Image Milnor number

If $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is $\mathcal{A}$-finite and (n, p) are nice dimensions, f has a *stable perturbation* $f_t : U_t \to V$, where U_t is a union of disjoint contractible neighbourhoods of the points of S. When $n \geq p$, or $n = p - 1$, the discriminant (image), $D(f_t) \simeq \bigvee S^{p-1}$. The number of spheres is the *discriminant Milnor number* $\mu_\Delta(f)$ (resp. the image Milnor number $\mu_I(f)$). In Mather's nice dimensions, $\mu_\Delta(f)$ and conjecturally $\mu_I(f)$ satisfy

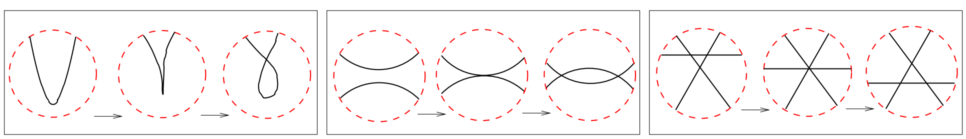
$$\mu_\Delta(f) \geq \mathcal{A}_e - \text{codim}(f) \quad (n \geq p)$$

$$\mu_I(f) \geq \mathcal{A}_e - \text{codim}(f) \quad (n = p - 1)$$

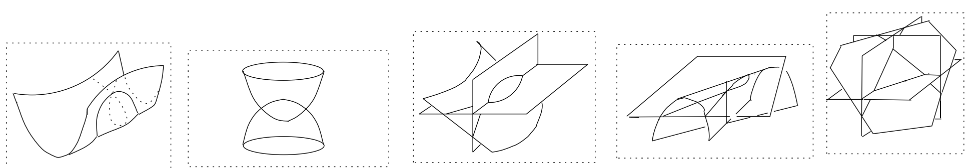
with equality in case f is $\mathcal{A}$-equivalent to a weighted homogeneous germ (see [dJvS91], [DM91],[Mon15]) .

When $\mu_I = 1$, you can often draw a good real picture, showing the vanishing homology in the right dimension (and therefore homotopy-equivalent to the complex image or discriminant). Examples:

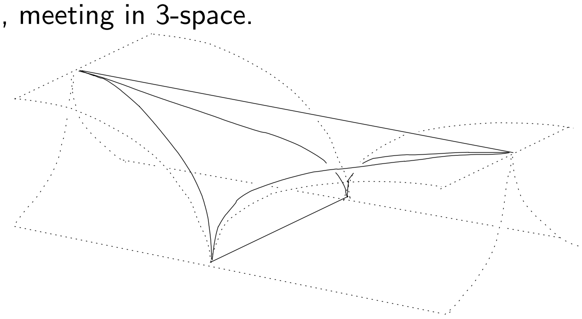
1. The Reidemeister moves, the singularities through which one generic planar projection of a knot must pass as it is deformed to another.



2. The Goryunov moves: images of stable perturbations of $\mathcal{A}_e$-codim1 germs from 2-space to 3-space.



3. The discriminant of a stable perturbation of a bi-germ consisting of two 1-parameter trivial unfoldings of a Whitney cusp, meeting in 3-space.



In each case the image has the homotopy type of a single sphere. But the way the homotopy-sphere is created is very different from one case to the next.

Here is how to describe it it. Let $f : X \to Y$ be surjective. Key notions:

1. Multiple point spaces,

$$D^k(f) = \text{closure}\{(x_1,\dots,x_k) : \ x_i \neq x_j \text{ if } i \neq j, \ f(x_i) = f(x_j) \text{for all } i,j\} \tag{1.1}$$

   These come with

   - S_k-action, and
   - projections $\pi^k : D^k(f) \to D^{k-1}(f), \quad \pi^k(x_1,\dots,x_k) = (x_1,\dots,x_{k-1})$.

   We denote the image of $D^k(f)$ in $D^j(f)$, $j < k$, by $D^k_j(f)$; the image of $D^k(f)$ in X is $D^k_1(f)$.

2. Alternating homology ([Gor95]). Suppose $f : X \to Y$ is surjective. With $C_\bullet(D^k(f))$ the usual singular chain complex, define

$$C^{\mathsf{Alt}}_j(D^k(f)) = \{c \in C_j(D^k(f)) : \sigma_\#(c) = \text{sign } \sigma \ \text{ for all } \sigma \in S_k\}.$$

   This gives a subcomplex, as $\partial(C^{\mathsf{Alt}}_j) \subset C^{\mathsf{Alt}}_{j-1}$, so we have *alternating homology*

$$H^{\mathsf{Alt}}_j(D^k(f)).$$

In fact we have a double complex: on $C^{\mathsf{Alt}}_j(D^k(f))$, $\pi^{k-1}_\# \circ \pi^k_\# = 0$; for

$$\pi^{k-1}_\# \circ \pi^k_\# = \pi^{k-1}_\# \circ \pi^k_\# \circ (k,k-1)_\#,$$

and on alternating chains $(k,k-1)_\#$ is multiplication by -1. And by same argument, $f_\# \circ \pi^2_\# = 0$. So

$$C^{\mathsf{Alt}}_\bullet(D^\bullet(f)) \text{ is a double complex}$$

The relevance to the homology of the image can be seen from two examples:

Example 1: let $c^2_j \in Z^{\mathsf{Alt}}_j(D^2(f))$.

$$\begin{tikzcd}
D^2(f) & 0 & c^2_j \arrow[l, "\partial"'] \arrow[d] & & \\
X & 0 & \pi^2_\#(c^2_j) \arrow[l] \arrow[d, "f_\#"] & c^1_{j+1} \arrow[l, "\partial"', "\exists"] \arrow[d] & \text{e.g. if } H_j(X) = 0 \\
Y & & f_\#\pi^2_\#(c^2_j) = 0 & f_\#(c^1_{j+1}) \arrow[l, "\partial"'] &
\end{tikzcd}$$

Because $f_\# \circ \pi^2_\# = 0$ on alternating chains, $f_\#(c^1_{j+1})$ is a cycle in Y. So from an alternating j-cycle c^2_j in $D^2(f)$, we get a $j+1$ cycle on Y – provided $\pi^2_\#(c^2_j)$ is a boundary in X, i.e. provided $\pi^2_*[c^2_j] = 0$ in $H_j(X)$. Note that

- If $c^2_j = \pi^{k+1}_\#(c^3_j)$ for $c^3_j \in C^{\mathsf{Alt}}_j(D^{k+1}(f))$ then $\pi^k_\#(c^2_j) = 0$.

- If $c_j^2 = \partial c_{j+1}^2$ for some $c_{j+1}^2 \in C_{j+1}^{\mathsf{Alt}}(D^2(f))$, then can take $c_{j+1}^1 = \pi_{\#}^2(c_{j+1}^2)$ so the homology class we get in $H_{j+1}(Y)$ is zero.

So we are really interested in

$$\frac{\ker \pi_*^2 : H_j^{\mathsf{Alt}}(D^2(f)) \to H_j(X)}{\mathsf{im}\, \pi_*^3 : H_j^{\mathsf{Alt}}(D^3(f)) \to H_j^{\mathsf{Alt}}(D^2(f))}$$

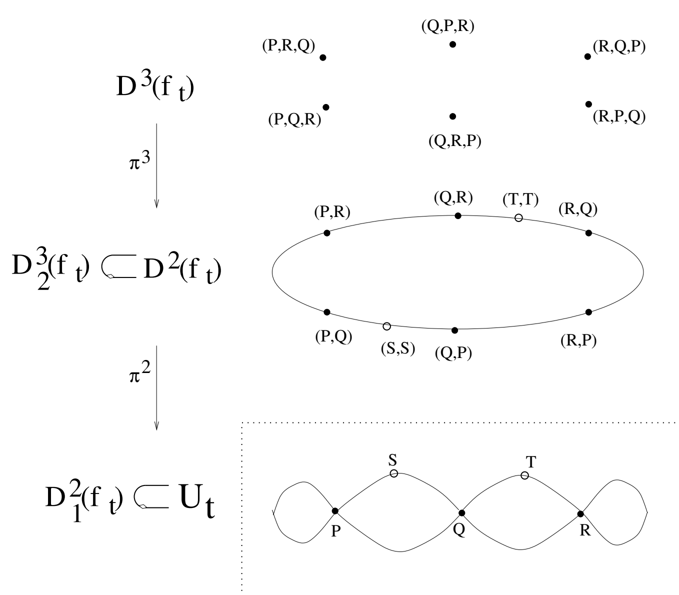
i.e. in *the homology of the vertical complex* $(H_j^{\mathsf{Alt}}(D^\bullet(f_t)), \pi_*^\bullet)$.

Example 2: let $c_j^3 \in Z_j(D^3(f))$.

$$\begin{tikzcd}
D^3(f) & 0 & c_j^3 \arrow[l, "\partial"'] \arrow[d] & & & \\
D^2(f) & 0 & \pi_{\#}^3(c_j^3) \arrow[l, "\partial"'] \arrow[d] & c_{j+1}^2 \arrow[l, "\partial", "\exists"'] \arrow[d] & & \text{provided } \pi_*^3[c_j^3] = 0 \in H_j^{\mathsf{Alt}}(D^2(f)) \\
X & & 0 & \pi_{\#}^2(c_{j+1}^2) \arrow[l, "\partial"'] \arrow[d] & c_{j+2}^1 \arrow[l, "\partial", "\exists"'] \arrow[d] & \text{provided } \pi_*^2[c_{j+1}^2] = 0 \in H_{j+1}(X) \\
Y & & & 0 & f_{\#}(c_{j+2}^1) \arrow[l, "\partial"'] &
\end{tikzcd}$$

Here, a j-dimensional homology class leads to a $j+2$-dimensional class in Y, provided certain homology classes vanish. Etc.

Example 3: Good real stable perturbation f_t of germ f of type H_2, $f(x,y) = (x, y^3, xy + y^5)$. Here there is one triple point, with preimages P, Q, R, and two cross-cap points S and T. The $\mathbb{Z}_2$-invariant points (S,S) and (T,T) lie in $D^2(f_t)$ (see (1.1)).



2 Image computing spectral sequence

Lurking behind this is the Image Computing Spectral Sequence. To calculate the homology of the image, begin with the double array

$$
\begin{array}{ccccc}
\downarrow & \downarrow & & \downarrow & \\
H_0^{\mathsf{Alt}}(D^p) & H_1^{\mathsf{Alt}}(D^p) & \cdots & H_r^{\mathsf{Alt}}(D^p(f)) & \cdots \\
\downarrow & \downarrow & & \downarrow & \\
H_0^{\mathsf{Alt}}(D^{p-1}) & H_1^{\mathsf{Alt}}(D^{p-1}) & \cdots & H_r^{\mathsf{Alt}}(D^{p-1}) & \cdots \\
\downarrow & \downarrow & & \downarrow & \\
\vdots & \vdots & \cdots & \vdots & \cdots \\
 & & \cdots & & \cdots \\
\downarrow & \downarrow & & \downarrow & \\
H_0(X) & H_1(X) & \cdots & H_r(X) & \cdots \\
\downarrow & \downarrow & & \downarrow & \\
0 & 0 & \cdots & 0 & \cdots
\end{array}
\tag{2.1}
$$

and zig-zag your way down to $H_q(Y)$.

Fortunately, in a stable perturbation of an $\mathcal{A}$-finite mono-germ this simplifies greatly:

Theorem 2.1. (K. Houston, [Hou97]) *The alternating homology of the multiple point spaces of a stable perturbation of an $\mathcal{A}$-finite germ is concentrated in middle dimension.*

Thus in each row of the diagram, at most one group is non-zero, and every arrow either begins or ends at a 0.

Corollary 2.2. *If f_t is a stable perturbation of an $\mathcal{A}$-finite germ $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$, then*

$$H_n(f_t(U_t)) \simeq \bigoplus_{k \geq 1} H^{\mathit{Alt}}_{n-k+1}(D^k(f_t)).$$

So if f has image Milnor number 1, then just one of these groups is non-zero, and we can say that the vanishing homology "comes from double points", or "comes from triple points", etc. A main theme of this talk: how to find out which?

The case of germs of corank 1 is well understood.

We have explicit equations for the multiple point spaces $D^k(f)$ (see [MM89]). For $k \leq n+1$, if is f $\mathcal{A}$-finite then $D^k(f)$ is an ICIS of dimension $n-k+1$ for $k \leq n+1$ and $D^k(f_t)$ is a Milnor fibre.

An $\mathcal{A}_e$- codimension 1 corank 1 map-germ of multiplicity $\ell+1$ from $(\mathbb{C}^{2\ell-1}, 0) \to (\mathbb{C}^{2\ell}, 0)$ is equivalent to

$$f(u, v, x) = (u, v, x^{\ell+1} + \sum_{i=1}^{\ell-1} u_i x^i, x^{\ell+2} + \sum_{i=1}^{\ell-1} v_i x^i)$$

We can check easily that $D^k(f)$ is smooth for $2 \leq k \leq \ell$, $D^{\ell+1}(f)$ is (isomorphic to) a hypersurface germ with A_1 singularity (and has dim $\ell - 1$), and $D^k(f) = \emptyset$ for $k > \ell + 1$. By a theorem of Orlik and Solomon on Milnor fibrations of invariant germs, as $S_{\ell+1}$-representations

$$H_{\ell-1}(D^{\ell+1}(f_t); \mathbb{Q}) = J \otimes_{\mathbb{Q}} \text{1-dimensional sign representation}$$

where J is the jacobian algebra of the hypersurface singularity. For an A_1 singularity, the jacobian algebra is just $\mathbb{Q}$, on which $S_{\ell+1}$ acts trivially, so

$$H_{\ell-1}(D^{\ell+1}(f_t); \mathbb{Q}) = \text{1-dimensional sign representation}$$

and

$$H^{\mathsf{Alt}}_{\ell-1}(D^{\ell+1}(f_t); \mathbb{Q}) = H_{\ell-1}(D^{\ell+1}(f_t); \mathbb{Q})$$

is 1-dimensional. Thus the vanishing homology comes from $\ell + 1$-tuple points, that is, from the top multiple point space.

Guess: This is always the case: the vanishing cycle in the image of a stable perturbation f_t of a codimension 1 germ $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ comes from $D^k(f_t)$ where k is the greatest integer such that $D^k(f_0) \neq \emptyset$.

Remark 2.3. In what follows, we concentrate on the rational homology of all of the spaces we are interested in. When we replace integer coefficients in the alternating chain complex by rational coefficients, for any space X with an S_k action we have

$$H^{\mathsf{Alt}}_q(X; \mathbb{Q}) \simeq \{c \in H_q(X; \mathbb{Q}) : \ \sigma_*(c) = \mathsf{sign}(\sigma)c \text{ for all } \ \sigma \in S_k\}.$$

The comparable equality with integer coefficients is false. A good example in which to see this is the quotient map q from the unit disc to $\mathbb{RP}^2$, in which diametrically opposite points on the boundary of the disc are identified. One sees that

$$H^{\mathsf{Alt}}_0(D^2(q); \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z},$$

clearly not a subspace of $H_0(D^2(q); \mathbb{Z})$.

3 A Corank 2 $\mathcal{A}$-codimension 1 germ

$$f(x, y, a, b, c, d) = (x^2 + ay, xy + bx + cy, y^2 + dx, a, b, c, d)$$

is stable and has corank 2.

$$f(x, y, a, b, c) = (x^2 + ay, xy + bx + cy, y^2 + ax, a, b, c)$$

has $\mathcal{A}_e$-codimension 1. The conjectured equality between μ_I and $\mathcal{A}_e$-codimension holds here: $\mu_I = 1$. By Cor. 2.2,

$$1 = \mathsf{dim} H_5(X_t) = \mathsf{dim} H^{\mathsf{Alt}}_4(D^2(f_t)) + \mathsf{dim} H_3(D^3(f_t)). \tag{3.1}$$

We have equations for $D^2(f)$ and $D^3(f)$ but these are neither ICIS nor determinantal, so give no direct information on the homology of their smoothings $D^2(f_t)$ and $D^3(f_t)$). The representation of S_3 on $H_3(D^3(fg_t))$ splits into isotypal components

$$H_3(D^3(f_t)) = H^T_3 \oplus H^{\mathsf{Alt}}_3 \oplus H^\rho_3$$

where T means trivial and ρ is the irreducible 2-dim representation.

We look for information on the projection of $D^3(f_t)$ to the domain of f_t, which we denote by $D^3_1(f_t)$, and to the image, which we denote by $M_3(F_t)$. We get this information from a presentation of $f_*(\mathcal{O}_{\mathbb{C}^5}$ as $\mathcal{O}_{\mathbb{C}^6}$-module. *Macaulay2* gives us the presentation

$$\begin{pmatrix} Y^2 - XZ - abZ - bcY + atY & aY + cX + tY & aY + bZ \\ aY + cX + tY & -Z - ac & Y - bc \\ aY + bZ & Y - bc & -X - ab - bt \end{pmatrix}. \tag{3.2}$$

1. The image triple points $M_3(f)$ are defined by the second Fitting ideal $\mathrm{Fitt}_2(f_*(\mathcal{O}_{\mathbb{C}^5,0}))$, generated in this case by the 1×1 minors of the matrix,

$$(Y - bc, X + ab, Z - ac) \tag{3.3}$$

This defines a smooth space, so $M_3(f_t)$, as a deformation of $M_3(f)$, is contractible. There are no quadruple points, so $M_3(f_t)$ is the quotient of $D^3(f_t)$ by the action of S_3. Thus

$$0 = H_3(M_3(f_t)) = H_3^T(D^3(f_t)).$$

2. $D^3_1(f) \subset (\mathbb{C}^5, 0)$ is defined by the pull back of Fitt_2:

$$(y^2 + ya + xc + ac, xy - bc, x^2 + xa + yb + ab)$$

Fortunately this is an isolated codimension 2 Cohen-Macaulay singularity, and therefore by the Hilbert Burch theorem, it is defined by the maximal minors of a $k \times (k+1)$ matrix for some k. One finds that

$$f^*(\mathrm{Fitt}_2) = \mathrm{min}_2 \begin{pmatrix} -y & -c \\ x + a & -y - a \\ b & x \end{pmatrix}$$

Calculation shows "τ"$= 1$, so this is isomorphic to the unique singularity with "τ"$= 1$ in the table on page 22 of the paper [FKZ15] of Fruhbis and Zach, which gives

$$b_0(D^3_1(f_t)) = 1, \quad b_1 = 0, \quad b_2 = 1, \quad b_3 = 0. \tag{3.4}$$

Note that $D^3_1(f_t)$ is a smoothing of $D^3_1(f)$. One can see this by listing the local normal forms of the stable singularities of mappings $\mathbb{C}^5 \to \mathbb{C}^6$. The only one that has D^3_1 singular is the quadruple point, and in our case there are none, since the multiplicity of f is < 4.

3. $D^3(f_t) \to D^3_1(f_t)$ is a branched double cover:

$$(P, Q, R) \mapsto P \quad (P, R, Q) \mapsto P$$

It is branched at $P \in D^3_1$ with preimage of the form (P, Q, Q) or (P, P, P). The closure of the first type contains the second. We denote the closure of the locus of points of the first type by $D^3_{1,0}(f_t)$. It is the "shadow component" of $f_t^{-1}(f(\Sigma f_t))$, where Σf_t is the non-immersive locus of f_t. That is,

$$D^3_{1,0}(f_t) = \text{closure of } \left(f_t^{-1}(f_t(\Sigma f_t)) \smallsetminus \Sigma f_t\right).$$

Its ideal is the saturation, the limit as $k \to \infty$ of transporter ideals

$$I(f_t^{-1}(f_t(\Sigma f_t))) : I(\Sigma f_t)^k.$$

When $t = 0$, a *Macaulay 2* calculation finds that this ideal is

$$\mathsf{min}_2 \begin{pmatrix} a & b & x & y \\ -3y+a & x+a & -y-a & 3y-a+4c \end{pmatrix}.$$

This is Pinkham's example of a germ with isolated singularity whose versal base space is reducible. In the deformation induced by the deformation of f, the ideal becomes

$$\mathsf{min}_2 \begin{pmatrix} a & b & x & y+t \\ -3y+a+t & x+a & -y-a-t & 3y-a+4c-t \end{pmatrix}$$

This defines a smoothing of $D^3_{1,0}(f_0)$, over the Artin component of the base space (since it is given by the minors of a 2×4 matrix). It is well known that the only non-zero reduced Betti number is $\beta_2 = 1$.

4. $D^3_1(f_t)$ is quotient of $D^3(f_t)$ by the $\mathbb{Z}_2$-action generated by the transposition

$$(2,3)(P,Q,R) = (P,R,Q).$$

So $H_i(D^3_1(f_t))$ is the part of $H_i(D^3(f_t))$ invariant under $(2,3)_*$. Since $H_i^T(D^3(f_t)) = 0$ for $i > 0$, and on $H_i^{\mathsf{Alt}}(D^3(f_t))$ $(2,3)_*$ is multiplication by -1, the $(2,3)_*$ -invariant part of $H_i(D^3(f_t))$ is just the $(2,3)_*$-invariant part of $H_i^\rho(D^3(f_t))$, and thus isomorphic to the sum of copies of the subspace of the two-dimensional irrreducible representation ρ invariant under $(2,3)$. The $(2,3)$-invariant subspace of ρ is 1-dimensional. Thus,

$$h_i(D^3_1(f_t)) = \frac{1}{2} h_i^\rho(D^3(f_t)) \tag{3.5}$$

for $i > 1$. Hence, by (3.4),

$$h_1^\rho(D^3(f_t)) = 0, \quad h_2^\rho(D^3(f_t)) = 2, \quad h_3^\rho(D^3(f_t)) = 0. \tag{3.6}$$

On the other hand, as $D^3(f_t)$ is a branched cover of degree 2 of $D^3_1(f_t)$, branched along $D^3_{1,0}(f_t)$, it follows that

$$\chi(D^3(f_t)) = 2\chi(D^3_1(f_t)) - \chi(D^3_{1,0}(f_t)) = 2.$$

Putting this together with (3.6), we have

$$2 = \chi(D^3(f_t)) = 1 - \left(h_1^\rho + h_1^{\mathsf{Alt}}\right) + \left(h_2^\rho + h_2^{\mathsf{Alt}}\right) - \left(h_3^\rho + h_3^{\mathsf{Alt}}\right) = 1 - h_1^{\mathsf{Alt}} + h_2^{\mathsf{Alt}} + 2 - h_3^{\mathsf{Alt}}.$$

so

$$-1 = -h_1^{\mathsf{Alt}} + h_2^{\mathsf{Alt}} - h_3^{\mathsf{Alt}}. \tag{3.7}$$

By Houston's theorem in [Hou97], the alternating homology of $D^3(f_t)$ is concentrated in middle dimension, so $h_i^{\mathsf{Alt}}(D^3(f_t)) = 0$ for $i \neq 3$ and so from (3.7), $h_3^{\mathsf{Alt}}(D^3_1(f_t)) = 1$.

5. Hence, by (3.1), $H_4^{\mathsf{Alt}}(D^2(f_t)) = 0$. The vanishing homology of the image comes from triple points.

Exercise

Copy the diagram on page 3.

A. (i) Mark an alternating 0-cycle c_0^3 on $D^3(f_t)$, by assigning a coefficient of $+1$ or -1 to each one of the six points
(ii) Draw an alternating 1-chain c_1^2 on $D^2(f_t)$ such that $\partial c_1^2 = \pi^3_{\#}(c_0^3)$.
(iii) Draw a 2-chain c_2^1 on U_t such that $\partial c_2^1 = \pi^2_{\#}(c_1^2)$.
Then $f_{\#}(c_2^1)$ is a 2-cycle on image(f_t).

B. (i) Draw an alternating 1-cycle e_1^2 on $D^2(f_t)$.
(ii) Draw a 2-chain e_2^1 on U_t such that $\partial e_2^1 = \pi^2_{\#}(e_1^2)$.
Then $f_{t\#}(e_2^1)$ is a 2-cycle on image(f_t).

C (harder) show that $[f_{\#}(c_2^1)]$ and $[f_{\#}(e_2^1)]$ are a basis for H_2(image(f_t)).

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