Topology of Exceptional Orbit Hypersurfaces of Prehomogeneous Spaces

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Topology of Milnor Fibration, Link, and Complement

$f : \mathbb{C}^{n+1}, 0 \subset \mathbb{C}, 0,\ X_0 = f^{-1}(0)$ of dimension n:

Milnor Fibration: $\mathcal{X} = f^{-1}(S^1_\delta) \cap B_\varepsilon(0) \to S^1_\delta$, for $0 < \delta << \varepsilon$

- Uses Morse theory to determine topology
- structure of Milnor Fiber: $(n-1)$–connected and $\simeq$ bouquet of n–spheres ("compact model") ,
- algebraic formula for Milnor number: $\mu =$ dimension Milnor algebra
- Link $L(X_0) = X_0 \cap S^{2n+1}_\varepsilon$ $(n-2)$ -connected $(2n-1)$-dim compact manifold $\simeq$ boundary of closed Milnor fiber
- monodromy $+$ Wang sequence to relate link and Milnor fiber

Geometry of Milnor Fiber

- intersection pairing
- cohomology via relative deRham complex
- monodromy and Gauss-Manin connection
- Mixed Hodge Structure
- relation with deformation theory and resolutions

Nonisolated Singularities from Perspective of Isolated Singularities

Theorem (Kato-Matsumoto): If $\dim(\mathrm{sing}(X)) = k$, then Milnor fiber is $(n-k-1)$–connected.

Is the Milnor fiber $\simeq$ bouquet of spheres?

Depends on properties of $\Sigma = \mathrm{sing}(X)$ and the transverse types of f on Σ (Siersma, Pellikan, Tibar, Nemethi, Zaharia).

- Σ an ICIS of $\dim = 1$: If f has transverse type A_1 (Siersma) then Milnor fiber is homotopy equivalent to a bouquet of S^n 's and possibly one S^{n-1}.
- Σ an ICIS of $\dim = 2$ If f has transverse type A_1 off curve C in Σ, Milnor fiber is homotopy equivalent to a bouquet of S^n 's and possibly one S^{n-1} or S^{n-2} (Zaharia and Nemethi).
- Additional results for $\dim \Sigma \leq 2$ and different transverse types: (Siersma, Pellikan, Tibar, Nemethi, Zaharia, Van Straten, etc)

Motivation for Nonisolated Singularities

Nonisolated singularities arising as "nonlinear sections" of singularities defined by a holomorphic germ

$$f_0 : \mathbb{C}^n, 0 \longrightarrow \mathbb{C}^N, 0 \supset \mathcal{V}, 0 \tag{1}$$

(or more generally $f_0 : X, 0 \to \mathbb{C}^N, 0$ for analytic germ $X, 0$).
The pull-back variety $\mathcal{V}_0 = f_0^{-1}(\mathcal{V})$ is the "singularity" defined by f_0 under $\mathcal{K}_{\mathcal{V}}$-equivalence of f_0.

$$\begin{array}{ccccc} & & \mathbb{C}^n, 0 & \xrightarrow{f_0} & \mathbb{C}^N, 0 \\ & & \uparrow & & \uparrow \\ f_0^{-1}(\mathcal{V}) & = & \mathcal{V}_0, 0 & \longrightarrow & \mathcal{V}, 0 \end{array} \tag{2}$$

Examples: Discriminants, Bifurcation Sets, Hyperplane Arrangements, Nonlinear Arrangements, Matrix Singularities, Special Classes of Singularities (e.g. Gorenstein and Cohen-Macaulay singularities), Quiver Representation Discriminants.

Prehomogeneous Spaces

Let $\rho : G \to GL(V)$ be a complex representation of a (connected) complex linear algebraic group G with an open orbit $\mathcal{U}$. Then, V is called a prehomogeneous (vector) space (due to Sato).
The complement $\mathcal{E} = V \backslash \mathcal{U}$ is the variety of orbits of positive codimension, which we call exceptional orbit variety.

Sato and Kimura: Classification of prehomogeneous spaces arising from irreducible representations of semisimple algebraic groups (for applications to harmonic analysis); $\mathcal{E}$ called the "singular set".

Actions of $GL_n(\mathbb{C})$ on the spaces of $m \times m$ matrices which are general (under left multiplication), or symmetric or skew-symmetric (m even) via $B \cdot A = BAB^T$. The exceptional orbit varieties are determinant varieties: hypersurfaces defined by $\det : M \to \mathbb{C}$:

$\mathcal{D}_m^{sy}$ for $M = Sym_m$; $\mathcal{D}_m$ for $M = M_{m,m}$; and
$\mathcal{D}_m^{sk}$ for $M = Sk_m$, $(m = 2k)$ defined by $Pf : Sk_m \to \mathbb{C}$.

Equidimensional Representations: $\dim G = \dim V$.
Then, $\mathcal{E}$ is a hypersurface.
Examples:

i) Reductive groups: include quivers of finite representation type. The exceptional orbit variety is called the the "discriminant". These are linear free divisors (Buchweitz and Mond)

ii) Solvable linear algebraic groups: *block representations* criteria for exceptional orbit varieties being free or free* divisors. Examples: (modified) Cholesky factorizations (D'and B. Pike)

iii) General linear algebraic groups formed as extensions of reductive groups by solvable linear algebraic groups: block representations yielding exceptional orbit varieties free or free* divisors.

Exceptional orbit varieties defined by block representations of solvable linear algebraic groups arising from modified Cholesky factorizations.

$\mathcal{E}$	Defining Equation for $\mathcal{E}$
$\mathcal{E}_m^{sy}$	$\prod_{k=1}^{m} \det(A^{(k)})$
$\mathcal{E}_m$	$\prod_{k=1}^{m} \det(A^{(k)}) \cdot \prod_{k=1}^{m-1} \det(\hat{A}^{(k)})$
$\mathcal{E}_{m-1,m}$	$\prod_{k=1}^{m-1} \det(A^{(k)}) \cdot \prod_{k=1}^{m-1} \det(\hat{A}^{(k)})$
$\mathcal{E}_m^{sk}$	$\prod_{k=1}^{m-2} \det\left(\hat{\hat{A}}^{(k)}\right) \cdot \prod_{k=2}^{m} \mathrm{Pf}_{\{\epsilon(k),\ldots,k\}}(A)$

Examples of Exceptional Orbit Varieties

- $\mathcal{E}_2^{sy}$: $x \cdot (xz - y^2)$
- $\mathcal{E}_3^{sy}$: $x \cdot (xw - y^2) \cdot (xu^2 + vy^2 + wz^2 - xwv - 2zyu)$

$$\begin{pmatrix} x & y & z \\ y & w & u \\ z & u & v \end{pmatrix}$$

- $\mathcal{E}_2$: $xy \cdot (xw - yz)$
- $\mathcal{E}_{2,3}$: $xy \cdot (xv - yu) \cdot (yw - zv)$
- D_4 quiver discriminant: $(xw - zu) \cdot (xv - yu) \cdot (yw - zv)$
- Cholesky factorization: $x \cdot (xw - yv + zu)$

$$\begin{pmatrix} 0 & x & y & z \\ -x & 0 & u & v \\ -y & -u & 0 & w \\ -z & -v & -w & 0 \end{pmatrix}$$

i) reversing the usual approach: begin with the complement and deduce the topology of the link and Milnor fiber.

ii) replacing the local Milnor fiber by a global Milnor fiber, which is a smooth affine hypersurface that has a "model complex geometry" resulting from the transitive action of an associated linear algebraic group, yielding as a deformation retract a compact submanifold;

iii) using the relation between the two algebraic group actions and the topology of maximal compact subgroups to deduce the cohomological triviality of an associated fibration of the groups;

iv) tools: Hopf structure theorem, Cartan's results on classical symmetric spaces, Wang sequence, Leray-Hirsch theorem, homotopy long exact sequence of fibration, Bott periodicity theorem;

v) using the preceding to determine the topology: (co)homology and homotopy groups of the Milnor fiber, link, and complement.

A *special prehomogeneous space* is a prehomogeneous space with $\mathcal{E}$ a hypersurface.

Lemma 1

The Milnor fibration of $(\mathcal{E}, 0)$ is diffeomorphic to the global Milnor fibration $f|E : E \to S^1$, where $E = f^{-1}(S^1)$, with fiber $F = f^{-1}(1)$. This is the restriction of the fibration $f : V \backslash \mathcal{E} \to \mathbb{C}^*$ to $S^1 \subset \mathbb{C}^*$, and the inclusion $E \subset V \backslash \mathcal{E}$ is a homotopy equivalence.

Lemma 2

If G and the isotropy subgroup H of $v_0 \in \mathcal{U}$ have maximal compact subgroups K, resp. L, then $V \backslash \mathcal{E}$ is homotopy equivalent to K/L. Hence,

$$H^*(V \backslash \mathcal{E}) \simeq H^*(K/L).$$

For a special prehomogeneous space $\rho : G \to \mathrm{GL}(V)$ with $\mathcal{E}$ the exceptional orbit hypersurface, let $h = 0$ be an irreducible defining equation for $\mathcal{E}$. Then, G acts on $\mathbb{C} \simeq \mathbb{C} < h >$ with character χ_0. Let $G' = \ker(\chi_0)$ and G'_0 the connected component of G'.

Lemma In the preceding situation, χ_0 is non-trivial and induces by a lifting an exact sequence

$$1 \longrightarrow G'_0 \longrightarrow G \xrightarrow{\chi} \mathbb{C}^* \longrightarrow 1 \qquad (3)$$

where G' and G'_0 are linear algebraic groups with $\dim_{\mathbb{C}} G'_0 = \dim_{\mathbb{C}} G - 1$ and $\mathrm{rank}(G'_0) = \mathrm{rank}(G) - 1$. Moreover, if G is reductive, respectively solvable, then so is G'_0 reductive, respectively solvable.

We can replace the linear algebraic groups by their maximal compact subgroups and obtain the exact rows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G_0' & \longrightarrow & G & \xrightarrow{\chi} & \mathbb{C}^* & \longrightarrow & 1 \\ & & i' \uparrow & & i \uparrow & & i_0 \uparrow & & \\ 1 & \longrightarrow & K_0' & \longrightarrow & K & \xrightarrow{\chi'} & S^1 & \longrightarrow & 1 \end{array} \qquad (4)$$

The following hold:

- **i)** each vertical arrow is a homotopy equivalence;
- **ii)** G_0' acts transitively on the global Milnor fiber F; and
- **iii)** By a "numerical criterion", the fibration $K_0' \hookrightarrow K \to S^1$ is cohomologically trivial.

Proposition 3
For a prehomogeneous space defined by the representation $\rho : G \to \mathrm{GL}(V)$ with exceptional orbit variety $\mathcal{E}$ a hypersurface, there is a connected codimension one algebraic subgroup G_0' of G which acts transitively on the global Milnor fiber F, with isotropy subgroup H' so that (F, G_0', H') defines a model complex geometry on F (in the sense of Thurston).

- **i)** In the case of determinantal hypersurfaces, this model is simply connected, and
- **ii)** in the equidimensional case, G_0' is a finite regular covering space of F with group of covering transformations H'.

Proposition

For the fibration $F \hookrightarrow E \to S^1$, with F connected and $\mathbf{k}$ a field of characteristic 0, the following are equivalent.

i) The fibration is cohomologically trivial (i.e. monodromy acts trivially on $H^*(F;\mathbf{k})$).

ii) there is an isomorphism of graded vector spaces.

$$\begin{aligned} H^*(E;\mathbf{k}) &\simeq \Lambda^*\mathbf{k}\langle s_1\rangle \otimes H^*(F;\mathbf{k}) \\ &\simeq H^*(F;\mathbf{k}) \ \oplus\ \mathbf{k}\langle s_1\rangle \otimes H^*(F;\mathbf{k})\,. \end{aligned} \tag{5}$$

iii)

$$\dim_{\mathbf{k}} H^*(E;\mathbf{k}) = 2\dim_{\mathbf{k}} H^*(F;\mathbf{k})\,. \tag{6}$$

Moreover, if the preceding hold then (5) is an isomorphism of graded $\Lambda^*\mathbf{k}\langle s_1\rangle$-modules, where the exterior algebra $\Lambda^*\mathbf{k}\langle s_1\rangle$ is on one generator s_1 of degree 1.

Topology of the exceptional orbit variety $\mathcal{E}$

Proposition Suppose $\rho : G \to \mathrm{GL}(V)$ is a representation which belongs to the special class of prehomogeneous spaces.

i) *Cohomology of Milnor fiber F:*
If the global Milnor fibration is cohomologically trivial, then

$$H^*(F;\mathbf{k}) \simeq H^*(E;\mathbf{k})/\left(\mathbf{k}\langle s_1\rangle \smile H^*(E;\mathbf{k})\right)$$

where as graded vector spaces,

$$H^*(E;\mathbf{k}) \simeq H^*(F;\mathbf{k}) \oplus \left(\mathbf{k}\langle s_1\rangle \otimes H^*(F;\mathbf{k})\right) ;$$

ii) *Cohomology of the Complement $V\backslash\mathcal{E}$:*

$$H^*(V\backslash\mathcal{E};\mathbf{k}) \simeq H^*(K/L;\mathbf{k}) ;$$

iii) *Cohomology of the Link $L(\mathcal{E})$:*
If K/L is orientable, then as graded vector spaces

$$\widetilde{H}^*(L(\mathcal{E});\mathbf{k}) \simeq H^*\widetilde{(K/L;\mathbf{k})}\,[2N-2-\dim_{\mathbb{R}} K/L] ;$$

For G a connected compact Lie group, $H \subseteq G$ a closed subgroup, $char(\mathbf{k}) = 0$.

$$\text{Let} \quad r = \frac{rank(H)}{rank(G)} \qquad 0 \leq r \leq 1$$

The structure of $H^*(G/H; \mathbf{k})$

1) If $r = 0$, then H is finite. By the Hopf structure theorem

$$H^*(G/H; \mathbf{k}) \simeq H^*(G; \mathbf{k})^H \simeq H^*(G; \mathbf{k}) \simeq \Lambda^* \mathbf{k}\{e_1, \ldots, e_m\}$$

where $m = rank(G)$ and the e_j have odd degree.

2) If $r = 1$, then $rank(H) = rank(G)$, and G/H is a generalized flag manifold and $H^*(G/H; \mathbf{k}) \simeq \mathbf{k}[a_1, \ldots, a_m]/I$, a quotient of a polynomial algebra by an ideal I of relations on a set of the characteristic classes a_i (of even degree).

3) For $0 < r \approx \frac{1}{2} < 1$, $H^*(G/H; \mathbb{R})$ isomorphic to algebra of closed left invariant forms on $\mathbf{g}$, which annihilate $\mathbf{h}$ and are invariant under $Ad(H)$.

Milnor Fibers of the Determinant Varieties

Theorem: The Milnor fibers of the determinant varieties are homogeneous spaces homotopy equivalent to classical symmetric spaces.

Determinant Variety	Milnor Fiber F	Symmetric Space	$H^*(F, \mathbf{k})$
$\mathcal{D}_m^{sy}$ (m = 2k+1)	$SL_m(\mathbb{C})/SO_m(\mathbb{C})$	SU_m/SO_m	$\Lambda^*\mathbf{k}\{e_5, e_9, \ldots, e_{4k+1}\}$
$\mathcal{D}_m^{sy}$ (m = 2k)	$SL_m(\mathbb{C})/SO_m(\mathbb{C})$	SU_m/SO_m	$\Lambda^*\mathbf{k}\{e_5, e_9, \ldots, e_{4k-3}\} \cdot \{1, e_{2k}\}$
$\mathcal{D}_m$	$SL_m(\mathbb{C})$	SU_m	$\Lambda^*\mathbf{k}\{e_3, e_5, \ldots, e_{2m-1}\}$
$\mathcal{D}_m^{sk}, m = 2k$	$SL_{2k}(\mathbb{C})/Sp_k(\mathbb{C})$	SU_{2k}/Sp_k	$\Lambda^*\mathbf{k}\{e_5, e_9, \ldots, e_{4k-3}\}$

Theorem: The cohomology of the complements and links of determinant varieties are given by the following table, where for the link the cohomology $H^*(K/L, \mathbf{k})$ is upper truncated and shifted.

Determinant Variety	Complement $M\backslash\mathcal{D}$	$H^*(M\backslash\mathcal{D}, \mathbf{k}) \simeq H^*(K/L, \mathbf{k})$	Shift
$\mathcal{D}_m^{sy}$ (m = 2k+1)	$GL_m(\mathbb{C})/O_m(\mathbb{C}) \sim U_m/O_m(\mathbb{R})$	$\Lambda^*\mathbf{k}\langle e_1, e_5, \ldots, e_{2m-1}\rangle$	$\binom{m+1}{2} - 2$
$\mathcal{D}_m^{sy}$ (m = 2k)	$GL_m(\mathbb{C})/O_m(\mathbb{C}) \sim U_m/O_m(\mathbb{R})$	$\Lambda^*\mathbf{k}\langle e_1, e_5, \ldots, e_{2m-3}\rangle$	$\binom{m+1}{2}+$ m - 2
$\mathcal{D}_m$	$GL_m(\mathbb{C}) \sim U_m$	$\Lambda^*\mathbf{k}\langle e_1, e_3, \ldots, e_{2m-1}\rangle$	$m^2 - 2$
$\mathcal{D}_m^{sk}$ (m = 2k)	$GL_{2k}(\mathbb{C})/Sp_k(\mathbb{C}) \sim U_{2k}/Sp_k$	$\Lambda^*\mathbf{k}\langle e_1, e_5, \ldots, e_{2m-3}\rangle$	$\binom{m}{2} - 2$

Numerical Criterion for cohomological triviality of monodromy is satisfied for all cases except $\mathcal{D}_m^{sy}$ for m even.

Remark: The same method applies to the link and complement of the variety of singular $m \times n$ matrices $\mathcal{V}_{m,n} \subset M_{m,n}$ for $m > n$ (even though they are not complete intersections or do not have Milnor fibers).

Theorem: The cohomology of the complements and links of determinant varieties $\mathcal{V}_{m,n}$ are given by:

$$H^*(M_{m,n} \backslash \mathcal{V}_{m,n}, \mathbf{k}) \simeq \Lambda^* \mathbf{k}\{e_{2(m-n)+1}, e_{2(m-n)+3}, \ldots, e_{2m-1}\}$$

where for the link the cohomology $H^*(M_{m,n} \backslash \mathcal{V}_{m,n}, \mathbf{k})$ is upper truncated and shifted by $n^2 - 2$ (as a graded vector space).

This follows because $M_{m,n} \backslash \mathcal{V}_{m,n}$ is homotopy equivalent to the Stiefel manifold $V_n(\mathbb{C}^m)$ of ordered sets of n orthonormal vectors in $\mathbb{C}^m$. The cohomology of these have been computed by a combination of results involving Whitehead, Borel, and C. E. Miller.

Theorem: The homotopy groups of the Milnor fibers up to the end of the stable range are as follows.

i)

$$\pi_j(F_m) \simeq \pi_j(SU_m) \simeq \pi_j(\mathbb{SU}) \qquad \text{for } j < 2m$$

ii)

$$\pi_j(F_m^{sy}) \simeq \pi_j(SU_m/SO_m) \simeq \pi_j(\mathbb{SU}/\mathbb{SO}) \qquad \text{for } j < m-1$$

iii) for $m = 2k$

$$\pi_j(F_m^{sk}) \simeq \pi_j(SU_{2k}/Sp_k) \simeq \pi_j(\mathbb{SU}/\mathbb{S}p) \qquad \text{for } j < 4k-2$$

where the stable homotopy groups are given in Table.

$\pi_j(\mathbb{G}/\mathbb{H})$ $j =$	0	1	2	3	4	5	6	7	8	9
$\mathbb{SU}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$\mathbb{SU}/\mathbb{SO}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
$\mathbb{SU}/\mathbb{S}p$	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$

Equidimensional Case

For an equidimensional representation $\rho : G \to \mathrm{GL}(V)$ with G a connected linear algebraic group G, having maximal compact subgroup K.
For $x_0 \in \mathcal{U}$, an open orbit of ρ, the isotropy subgroup H is finite.
For a field $\mathbf{k}$ of characteristic 0, by the Hopf structure theorem,

$$H^*(K; \mathbf{k}) \;=\; \Lambda^* \mathbf{k}\langle s_1, s_2, \ldots, s_k \rangle \;. \tag{7}$$

where s_j are classes of odd degree q_j and $rank(K) = k$.

Topology in the Equidimensional Case

Theorem:
Topology of the complement:

$$H^*(V\backslash\mathcal{E};\mathbf{k}) \;\simeq\; H^*(K;\mathbf{k}) \;=\; \Lambda^*\mathbf{k}\langle s_1, s_2, \ldots, s_k\rangle\ . \tag{8}$$

In addition, $\pi_i(V\backslash\mathcal{E}) \simeq \pi_i(K)$ for $i > 1$; and there is a short exact sequence

$$1 \longrightarrow \pi_1(K) \longrightarrow \pi_1(V\backslash\mathcal{E}) \longrightarrow H \longrightarrow 1 \tag{9}$$

Topology of the Link: As graded vector spaces

$$\widetilde{H}_*(L(\mathcal{E});\mathbf{k}) \;\simeq\; \widetilde{H}^*(L(\mathcal{E});\mathbf{k}) \;\simeq\; \widetilde{\Lambda^*\mathbf{k}}\langle s_1, s_2, \ldots, s_k\rangle\,[2N-2-\dim_{\mathbb{R}}K]$$

where $N = \dim_{\mathbb{C}} V = \dim_{\mathbb{C}} G$

Topology in the Equidimensional Case (cont)

Topology of the Milnor Fiber:

The Milnor fibration is cohomologically trivial (by the numerical criterion) and

$$H^*(F; \mathbf{k}) \simeq \Lambda^* \mathbf{k} < e_2, \ldots, e_k > .$$

Here $e_j = i^*(s_j)$, where $i : F \hookrightarrow V \backslash \mathcal{E}$ is the inclusion.
Moreover, the homotopy groups of F are given by $\pi_j(F) \simeq \pi_j(G)$ for $j \geq 2$; and there is the exact sequence

$$1 \longrightarrow \pi_1(G_0') \longrightarrow \pi_1(F) \longrightarrow H \longrightarrow 1 \qquad (10)$$

where H is the isotropy group of G_0' for a point in F and $\pi_1(G_0')$ is in the exact sequence

$$1 \longrightarrow \pi_1(G_0') \longrightarrow \pi_1(G) \longrightarrow \mathbb{Z} \longrightarrow 1 \qquad (11)$$

Special Case: G solvable

Theorem (D'and B. Pike):

i) $V \backslash \mathcal{E}$ is a $K(\pi, 1)$ with π a finite extension of $\mathbb{Z}^k$ by the finite isotropy group H of a point in $\mathcal{U}$; and

$$H^*(V \backslash \mathcal{E}; \mathbf{k}) \;=\; \Lambda^* \mathbf{k} \langle s_1, s_2, \ldots, s_k \rangle \;.$$

where each s_j is of degree one.

ii) The Milnor fiber F is a $K(\pi, 1)$ with π given by the exact sequence

$$1 \longrightarrow \pi_1(F) \longrightarrow \pi_1(V \backslash \mathcal{E}) \longrightarrow \mathbb{Z} \longrightarrow 1 \quad (12)$$

iii) For the (modified) Cholesky-factorizations, $\pi_1(V \backslash \mathcal{E}) \simeq \mathbb{Z}^k$

Discriminants of Quiver Representations of Finite Type

A quiver is a connected finite directed graph Γ with edges $e(\Gamma) = \{\ell_j\}$, vertices $v(\Gamma) = \{v_i\}$, where we denote the initial vertex for ℓ_j by $i(\ell_j)$ and end point by $e(\ell_j)$.
For a dimension vector $\mathbf{d}$, a *representation of the quiver* is formed by associating to each vertex v_i a finite dimensional complex vector space V_i of dimension d_i and to each edge ℓ_j a linear transformation $\varphi_j : V_{i(\ell_j)} \to V_{e(\ell_j)}$.
Quiver representation space $V \simeq \prod_{\ell_j \in e(\Gamma)} \mathrm{Hom}(V_{i(\ell_j)}, V_{e(\ell_j)})$. The group $\tilde{G} = \prod_{v_i \in v(\Gamma)} \mathrm{GL}(V_i)$ acts on V by

$$\{\psi_i\} \cdot \{\varphi_j\} \; = \; \{\psi_{e(\ell_j)} \circ \varphi_j \circ \psi^{-1}_{i(\ell_j)}\} \qquad \textit{for} \quad \{\psi_i\} \in \tilde{G},\ \{\varphi_j\} \in V$$

Theorem(Gabriel) The quiver representations of finite type are given by Γ a Dynkin diagram of type A, D, or E and $\mathbf{d}$ a positive Schur root corresponding to the Dynkin diagram.

Theorem(Buchweitz and Mond) The representation of $G = \tilde{G}/\mathbb{C}^*$ on V gives an equidimensional representation for which the "quiver discriminant" $\mathcal{D}_{(\Gamma,\mathbf{d})}$ $(= \mathcal{E})$ is a linear free divisor.
For $\tilde{K}$ the maximal compact subgroup of $\tilde{G}$,

$$\Lambda^*(\Gamma, \mathbf{d}) \overset{def}{=} H^*(\tilde{K}; \mathbf{k}) \simeq \otimes_{v_i \in v(\Gamma)} \Lambda^* \mathbf{k}\langle s_1^{(i)}, \ldots, s_{d_i}^{(i)} \rangle \tag{13}$$

Theorem Let $F_{(\Gamma,\mathbf{d})}$ denote the Milnor fiber of the discriminant $\mathcal{D}_{(\Gamma,\mathbf{d})}$, and $L(\mathcal{D}_{(\Gamma,\mathbf{d})})$ the link. Then,

$$H^*(F_{(\Gamma,\mathbf{d})}; \mathbf{k}) \simeq \Lambda^*(\Gamma, \mathbf{d})/(s_1, s_2) \cdot \Lambda^*(\Gamma, \mathbf{d}) \tag{14}$$

which is the exterior algebra on the generators of (13) but with two degree 1 generators removed. Also,

$$\widetilde{H}^*(L(\mathcal{D}_{(\Gamma,\mathbf{d})}); \mathbf{k}) \simeq \widetilde{\Lambda^*(\Gamma, \mathbf{d})}/(s_1 \cdot \widetilde{\Lambda^*(\Gamma, \mathbf{d})})[\dim_{\mathbb{C}} V - 2] \tag{15}$$

which is the exterior algebra on the generators of (13) with one degree 1 generator removed, then truncated in the top degree, and then shifted by degree $\dim_{\mathbb{C}} V - 2$.

Further Directions on the Geometry of the Milnor fibers

- Applying a theorem of Mutsuo Oka to obtain the topology for a formal linear combination of functions defining exceptional orbit varieties for which compact models of Milnor fibers are joins of compact manifolds.
- Determine a cell decomposition of the Milnor fibers - analog of Schubert decomposition, e.g. using "Cartan model for symmetric spaces"with Iwasawa decomposition, with closures of cells as suspensions of varieties.
- Using results from K-theory to determine the dual cohomology classes of the closures.
- Determine the image of the canonical subalgebra in the cohomology of Milnor fibers for matrix singularities.
- Determine further properties of the Milnor fibers and links using e.g. mixed Hodge structure and intersection homology.

Reference:

J. Damon, *Topology of Exceptional Orbit Hypersurfaces of Prehomogeneous Spaces*, to appear Journal of Topology, prepublication pdf available on Oxford Univ. Press, Journal of Topology, web page or as a preprint on the ArXiv