The cr-invariant and the configurations of generic curves for surfaces in $\mathbb{R}^4$

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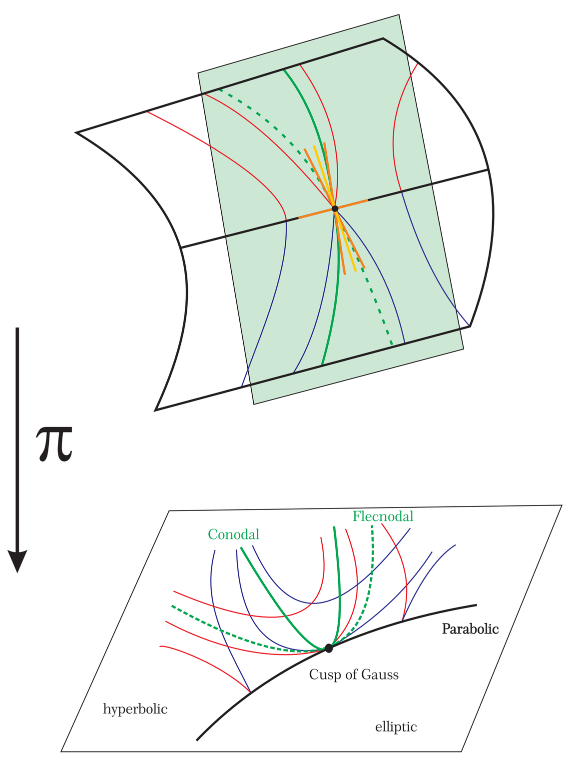
1. Uribe-Vargas' cr-invariant

Uribe-Vargas introduced a cross-ratio (cr-**invariant**) at a cusp of Gauss on a surface in $\mathbb{R}^3$ and related it to the modulus in the normal form of the 4-jet of a parameterisation the surface up to projective equivalence, which is given by ([Platanova, 81])

$$z = \frac{x^2}{2} - xy^2 + \lambda y^4, \quad \lambda \neq 0, \ \frac{1}{2}.$$

Uribe-Vargas considered the lift of some curves on M to PT^*M. The curves are the parabolic set, the flecnodal curve and the conodal curve (see Figure bellow). The tangent lines to their lift lie in the same contact plane at the cusp of Gauss and adding the vertical line gives 4 lines in that plane. The cross-ratio ρ of these lines is the cr-**invariant**. Uribe-Vargas showed that

$$\rho = 2\lambda.$$



2. Objective

For surfaces in $\mathbb{R}^4$, the $P_3(c)$-points have similar behavior to that of the cusps of Gauss on surfaces in $\mathbb{R}^3$. Our aim is to introduce cr-**invariants** at $P_3(c)$-points and relate them to the moduli in the 4-jet of a parametrisation of the surface up to projective equivalence. We present such curves and we list the possible configurations that occur on parabolic, S_2, B_2, flecnodal and **asymptotic curves** at $P_3(c)$-points through of the **cr-invariant**.

3. Asymptotic Curves

The **asymptotic curves** on a surface M in $\mathbb{R}^4$ are solutions of the BDE

$$\Omega(x,y,p) = (am-bl)p^2 + (an-cl)p + (bn-cm) = 0,$$

where a, b, c and l, m, n are coefficients of 2^{nd} fundamental form. The discriminant of the BDE is the zero set of the function

$$\delta = (an-cl)^2 - 4(am-bl)(bn-cm).$$

The BDE determins two 2 (resp. 1 or 0) asymptotic directions at each point on M. The point is called **hyperbolic** (resp. **parabolic** or **elliptic point**) if $\delta > 0$ (resp. $= 0$ or < 0).

4. $P_3(c)$-points and folded singularity

The asymptotic directions capture the contact of M with lines. This contact is determined by the singularities of the members of the family the ortogonal projections

$$P : M \times S^3 \to TS^3$$

where $P(p,u) = (u, p - \langle p, v\rangle v)$. For u fixed, the projection P_u can be viewed locally as a map-germ $P_u : \mathbb{R}^2, 0 \to \mathbb{R}^3, 0$.

The singularity of P_u at p is worse than a cross-cap if and only if u is an asymptotic direction at p.

For generic points on Δ, the **asymptotic curves** is a family of cusps tracing the curve Δ. At isolated points on Δ the germ P_u is $\mathcal{A}_e$-equivalent the $P_3(c)$-singularity

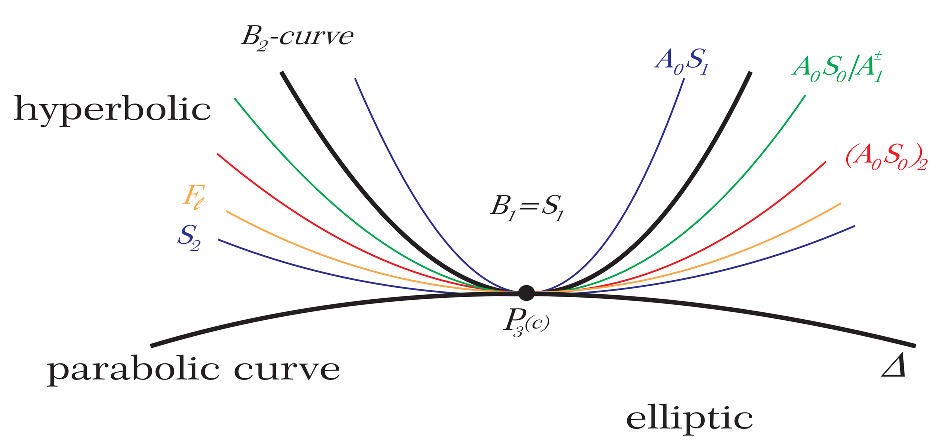
$$(x, xy^2 + cy^4, xy + y^3), \quad c \neq 0, \frac{1}{2}, 1, \frac{3}{2}.$$

The $P_3(c)$-points are precisely those where the **asymptotic curves** have a *folded singularity*.

The 2-jet of the curve of the inflections of the asymptotic curves at $P_3(c)$-point is given by $j^2\Omega = j^2(\Omega_y + p\Omega_x) = 0$. We denoted it by F_l.

5. Special curves at a $P_3(c)$-point

Theorem 1. *At a $P_3(c)$-point passe the following curves: the F_l curve and the curves obtained from the local and multi-local singularities of P_u. All these curves generically have tangency of order 2.*



6. Projective Classification

Theorem 2 (D-S, Kabata). *[2] The 4-jet of a parametrisation of a surface at $P_3(c)$-point is equivalent by projective transformations to the normal form*

$$(z, w) = (x^2 + xy^2 + \alpha y^4, xy + \beta y^3 + \phi),$$

where $6\beta^2 + 4\alpha - 15\beta + 5 \neq 0$, $\alpha \neq 0, 1/2, 1, 3/2$ and ϕ is a polynomial of degree 4

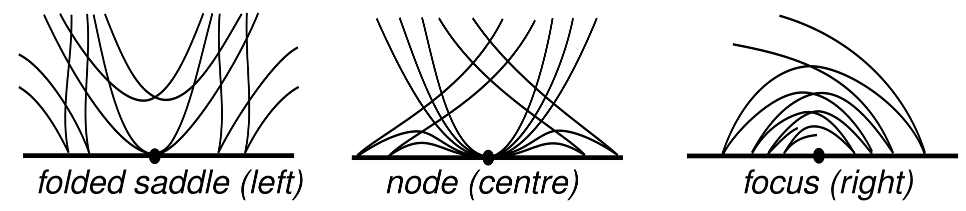
7. The cr-invariants at $P_3(c)$-points

The tangents lines of the Legendrian curves of Theorem 1 and the vertical line (contact element) at $P_3(c)$-point lie in the same contact plane in PT^*M. The cross-ratio of four of these lines at $P_3(c)$-point we call cr-**invariant**.

Theorem 3. *At a generic $P_3(c)$-point, three cross-ratios of the lines above allow to recover the projective invariants α and β in Theorem 2.*

8. The configuration of generic curves

Theorem 4. *At $P_3(c)$-points the* **asymptotic curves** *have a folded singularity if $\gamma = (-9\beta^2 - 6\alpha + (15/2)\beta) \neq 0, \frac{1}{16}$. The singularity is a folded saddle if $\gamma < 0$, a folded node if $0 < \gamma < \frac{1}{16}$ and folded focus if $\gamma > \frac{1}{16}$.*



The relative position of the curves Δ, B_2, S_2 and F_l is given by values of α and β. We parametrize the 2-jet of curves Δ, B_2, S_2 and F_l by $x = c_P \cdot y^2$, $x = c_B \cdot y^2$, $x = c_S \cdot y^2$ and $x = c_F \cdot y^2$, respectively.

Theorem 5. *At a $P_3(c)$-points there are 4 possibilities to the relative positions of the curves Δ, B_2, S_2, F_l:*

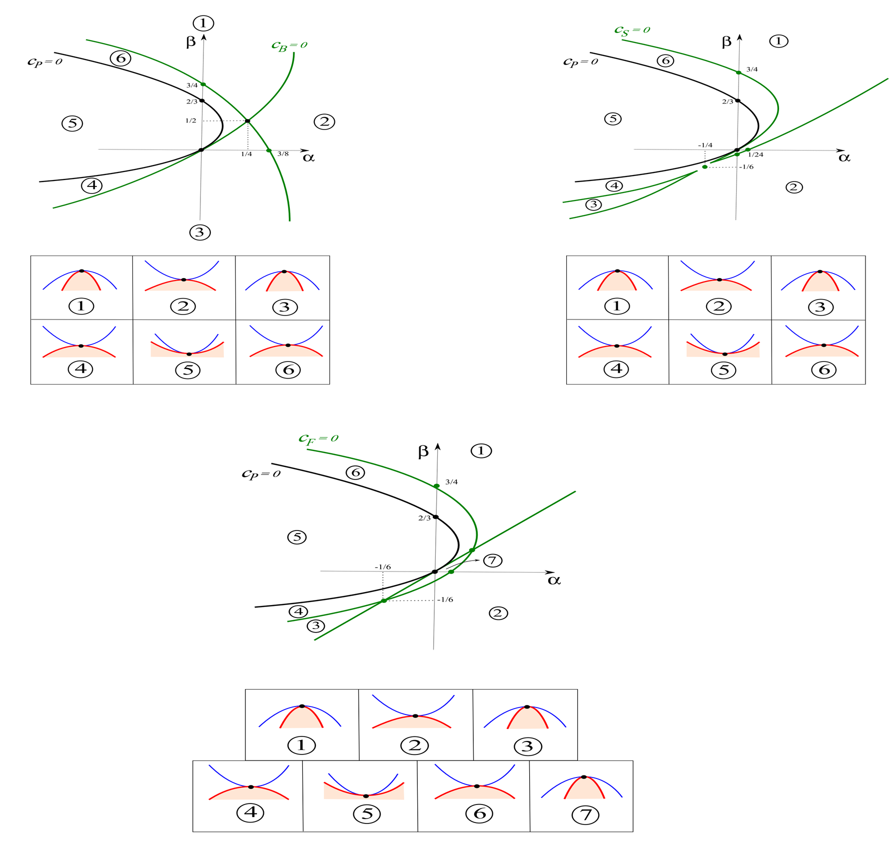
(i) If $\beta < 0$, then $c_P < c_B < c_F < c_S$

(ii) If $0 < \beta < 1/6$, then $c_P < c_B < c_S < c_F$

(iii) If $1/6 < \beta < 1/3$, then $c_P < c_S < c_B < c_F$

(iv) If $\beta > 1/3$, then $c_P < c_S < c_F < c_B$.

Theorem 6. *The configurations of curves Δ and B_2 (resp. S_2 and flecnodal) at $P_3(c)$-points in functions of α and β are described in Figure bellow.*



References

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[2] J. L. Deolindo-Silva and Y. Kabata, Projective Classification of Jets in $\mathbb{P}^4$. Preprint, 2016.

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