Lipschitz Regularity and multiplicity of analytic sets

Jose Edson Sampaio
Universidade Federal do Ceará

July 28, 2016

Lipschitz regularity

Lipschitz regularity

Theorem (______ (2016))

Let $X \subset \mathbb{C}^n$ be a complex analytic set. If there is a bi-Lipschitz homeomorphism $h : (X, 0) \to (\mathbb{C}^d, 0)$, then $(X, 0)$ is smooth.

Multiplicity

Multiplicity

Theorem (Fernandes and _______ (2016))

The metric version of the Zariski's conjecture has a positive answer if, and only if, it has a positive answer for homogeneous algebraic sets.

Multiplicity

Theorem (Fernandes and _______ (2016))

The metric version of the Zariski's conjecture has a positive answer if, and only if, it has a positive answer for homogeneous algebraic sets.

Theorem (Fernandes and _______ (2016))

Let $f, g : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ *be two reduced analytic function-germs and* $h : (\mathbb{C}^3, V(f), 0) \to (\mathbb{C}^3, V(g), 0)$ *be a bi-Lipschitz homeomorphism. Then,* $m(V(f), 0) = m(V(g), 0)$.

Motivation

Zariski's conjectures

Zariski's conjectures

In 1971, O. Zariski proposed the following problem:

Zariski's conjectures

In 1971, O. Zariski proposed the following problem:

Problem A (Zariski's Conjecture)

Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two reduced analytic function-germs. If there is a homeomorphism $h : (\mathbb{C}^n, V(f), 0) \to (\mathbb{C}^n, V(g), 0)$, then is it true that $m(V(f), 0) = m(V(g), 0)$?

Differentiable invariance of multiplicity

Differentiable invariance of multiplicity

Theorem (Ephraim (1976))

If there is a homeomorphism $h : (\mathbb{C}^n, V(f), 0) \to (\mathbb{C}^n, V(g), 0)$ *such that* h *and* h^{-1} *are differentiable at origin, then* $m(V(f), 0) = m(V(g), 0)$.

Differentiable invariance of multiplicity

Theorem (Ephraim (1976))

If there is a homeomorphism $h : (\mathbb{C}^n, V(f), 0) \to (\mathbb{C}^n, V(g), 0)$ *such that* h *and* h^{-1} *are differentiable at origin, then* $m(V(f), 0) = m(V(g), 0)$.

Theorem (Trotman (1977))

If there is a homeomorphism $h : (\mathbb{C}^n, V(f), 0) \to (\mathbb{C}^n, V(g), 0)$ *such that* h *is* C^1*, then* $m(V(f), 0) = m(V(g), 0)$.

Invariance of multiplicity in codimension ≥ 1

Invariance of multiplicity in codimension ≥ 1

Theorem (Gau and Lipman (1983))

If there is a homeomorphism $h : (\mathbb{C}^n, X, 0) \to (\mathbb{C}^n, Y, 0)$ *such that* h *and* h^{-1} *are differentiable at origin, then* $m(X, 0) = m(Y, 0)$.

Invariance of multiplicity in codimension ≥ 1

Theorem (Gau and Lipman (1983))

If there is a homeomorphism $h : (\mathbb{C}^n, X, 0) \to (\mathbb{C}^n, Y, 0)$ such that h and h^{-1} are differentiable at origin, then $m(X,0) = m(Y,0)$.

Theorem (Comte (1998))

Given $(X,0)$ and $(Y,0)$ two complex analytic germs of $\mathbb{C}^n$ of dimension $d \leq n$, $M = \max(m(X,0), m(Y,0))$ and $h : (X,0) \to (Y,0)$ a bi-Lipschitz homeomorphism such that:

$$\frac{1}{C'}\|x-y\| \leq \|h(x) - hy)\| \leq C\|x-y\|, \quad \textit{for all } x, y \in X$$

and $C'C \leq (1+\frac{1}{M})^{\frac{1}{2d}}$, then $m(X,0) = m(Y,0)$.

Invariance of multiplicity of complex surfaces

Invariance of multiplicity of complex surfaces

Theorem (Saeki (1988) and Yau (1988))

Let $(X,0)$ and $(Y,0)$ be two complex analytic surfaces of $\mathbb{C}^3$. Suppose that X and Y are quasi-homogeneous with isolated singularity. If there is a homeomorphism $h:(\mathbb{C}^3,X,0)\to(\mathbb{C}^3,Y,0)$, then $m(X,0)=m(Y,0)$.

Invariance of multiplicity of complex surfaces

Theorem (Saeki (1988) and Yau (1988))

Let $(X,0)$ and $(Y,0)$ be two complex analytic surfaces of $\mathbb{C}^3$. Suppose that X and Y are quasi-homogeneous with isolated singularity. If there is a homeomorphism $h:(\mathbb{C}^3,X,0)\to(\mathbb{C}^3,Y,0)$, then $m(X,0)=m(Y,0)$.

Theorem (Pichon and Neumann (2016))

Let $(X,0)$ and $(Y,0)$ be two normal complex surfaces of $\mathbb{C}^n$. If there is a bi-Lipschitz homeomorphism $h:(X,0)\to(Y,0)$, then $m(X,0)=m(Y,0)$.

Metric version of the Zariski's conjecture

Metric version of the Zariski's conjecture

Conjecture A

Let $X, Y \subset \mathbb{C}^n$ be two complex analytic sets. If there is a bi-Lipschitz homeomorphism $h : (\mathbb{C}^n, X, 0) \to (\mathbb{C}^n, Y, 0)$, then $m(X, 0) = m(Y, 0)$.

Metric version of the Zariski's conjecture

Conjecture A

Let $X, Y \subset \mathbb{C}^n$ be two complex analytic sets. If there is a bi-Lipschitz homeomorphism $h : (\mathbb{C}^n, X, 0) \to (\mathbb{C}^n, Y, 0)$, then $m(X, 0) = m(Y, 0)$.

Conjecture AH

Let $X, Y \subset \mathbb{C}^n$ be two irreducible homogeneous complex algebraic sets. If there is a bi-Lipschitz homeomorphism $h : (\mathbb{C}^n, X, 0) \to (\mathbb{C}^n, Y, 0)$, then $m(X, 0) = m(Y, 0)$.

The conjectures A and AH are equivalents

The conjectures A and AH are equivalents

Theorem (Fernandes and _______ (2016))

The Conjecture A has a positive answer if, and only if, the Conjecture AH has a positive answer.

Regularity

Regularity

Theorem (Mumford (1961))

Topologically regular and normal complex surface is smooth.

Regularity

Theorem (Mumford (1961))

Topologically regular and normal complex surface is smooth.

Theorem (Prill (1967))

Topologically regular complex cone is a plane.

Regularity

Theorem (Mumford (1961))

Topologically regular and normal complex surface is smooth.

Theorem (Prill (1967))

Topologically regular complex cone is a plane.

Theorem (A'Campo (1973) and Lê (1973))

If there is a homeomorphism $h : (\mathbb{C}^n, V(f), 0) \to (\mathbb{C}^n, V(g), 0)$ *and* $m(V(f), 0) = 1$, *then* $m(V(g), 0) = 1$.

Regularity

Regularity

Theorem (Birbrair, Fernandes, Lê and _______ (2016))

Let $X \subset \mathbb{C}^n$ be a complex analytic set. If there is a subanalytic bi-Lipschitz homeomorphism $h : (X, 0) \to (\mathbb{C}^d, 0)$, then $(X, 0)$ is smooth.

Regularity

Theorem (Birbrair, Fernandes, Lê and ______ (2016))

Let $X \subset \mathbb{C}^n$ be a complex analytic set. If there is a subanalytic bi-Lipschitz homeomorphism $h : (X, 0) \to (\mathbb{C}^d, 0)$, then $(X, 0)$ is smooth.

Theorem (______ (2016))

Let $X \subset \mathbb{C}^n$ be a complex analytic set. If there is a bi-Lipschitz homeomorphism $h : (X, 0) \to (\mathbb{C}^d, 0)$, then $(X, 0)$ is smooth.

Preliminaries

Tangent cone

Tangent cone

Definition

We say that $v \in \mathbb{R}^n$ is a tangent vector of X at $x_0 \in \mathbb{R}^n$ if there are a sequence of points $\{x_i\} \subset X \setminus \{x_0\}$ tending to x_0 and sequence of positive real numbers $\{t_i\}$ such that

$$\lim_{i \to \infty} \frac{1}{t_i}(x_i - x_0) = v.$$

Tangent cone

Definition

We say that $v \in \mathbb{R}^n$ is a tangent vector of X at $x_0 \in \mathbb{R}^n$ if there are a sequence of points $\{x_i\} \subset X \setminus \{x_0\}$ tending to x_0 and sequence of positive real numbers $\{t_i\}$ such that

$$\lim_{i\to\infty} \frac{1}{t_i}(x_i - x_0) = v.$$

Definition

Let $C(X, x_0)$ denote the set of all tangent vectors of X at $x_0 \in \mathbb{R}^n$. We call $C(X, x_0)$ the **tangent cone** of X at x_0.

Lipschitz invariance of the tangent cone

Lipschitz invariance of the tangent cone

Theorem (Koike and Paunescu (2009))

Let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ *be a bi-Lipschitz homeomorphism. If* A *and* $h(A)$ *are subanalytics sets at* $0 \in \mathbb{R}^n$*, then* $\dim C(A, 0) = \dim C(h(A), 0)$.

Lipschitz invariance of the tangent cone

Theorem (Koike and Paunescu (2009))

Let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ *be a bi-Lipschitz homeomorphism. If* A *and* $h(A)$ *are subanalytics sets at* $0 \in \mathbb{R}^n$*, then*
$\dim C(A, 0) = \dim C(h(A), 0)$.

Theorem (______ (2016))

Let $X, Y \subset \mathbb{R}^n$ *be two germs of subanalytic subsets. If* $h : (X, 0) \to (Y, 0)$ *is a bi-Lipschitz homeomorphism, then there is a bi-Lipschitz homeomorphism* $dh : (C(X, 0), 0) \to (C(Y, 0), 0)$.

Lelong's numbers

Lelong's numbers

Definition

Consider the mapping $\rho : \mathbb{S}^{m-1} \times \mathbb{R}^+ \to \mathbb{R}^m$ given by $\rho(x, r) = rx$.

Lelong's numbers

Definition

Consider the mapping $\rho : \mathbb{S}^{m-1} \times \mathbb{R}^+ \to \mathbb{R}^m$ given by $\rho(x, r) = rx$.

Proposition/definition

Let $X \subset \mathbb{C}^n$ be a complex analytic set such that $0 \in X$. If $X_1, ..., X_r$ are the irreducible components of the tangent cone $C(X, 0)$, then for each X_j and for $x \in (X_j \cap \mathbb{S}^{2n-1}) \times \{0\}$ generic, the number of connected components of $\rho^{-1}(X \setminus \{0\}) \cap U_x$ is constant, where U_x is an open sufficiently small with $x \in U_x$. In this case, we define this number by $k_X(X_j)$.

Lelong's numbers

Definition

Consider the mapping $\rho : \mathbb{S}^{m-1} \times \mathbb{R}^+ \to \mathbb{R}^m$ given by $\rho(x, r) = rx$.

Proposition/definition

Let $X \subset \mathbb{C}^n$ be a complex analytic set such that $0 \in X$. If $X_1, ..., X_r$ are the irreducible components of the tangent cone $C(X, 0)$, then for each X_j and for $x \in (X_j \cap \mathbb{S}^{2n-1}) \times \{0\}$ generic, the number of connected components of $\rho^{-1}(X \setminus \{0\}) \cap U_x$ is constant, where U_x is an open sufficiently small with $x \in U_x$. In this case, we define this number by $k_X(X_j)$.

$k_X(X_j)$ is called the Lelong number of X_j (over X).

Invariance of Lelong's numbers

Invariance of Lelong's numbers

Theorem (Kurdika and Raby (1989))

The Lelong's numbers are invariants by analytic isomorphism.

Invariance of Lelong's numbers

Theorem (Kurdika and Raby (1989))

The Lelong's numbers are invariants by analytic isomorphism.

Theorem (Valette (2010))

The Lelong's numbers are invariants by subanalytic bi-Lipschitz homeomorphism.

Lipschitz invariance of the Lelong's numbers

Theorem (Fernandes and _______ (2016))

Let $X, Y \subset \mathbb{C}^n$ be germs of analytic subsets at $0 \in \mathbb{C}^n$ and let $X_1, \ldots, X_r$ and $Y_1, \ldots, Y_s$ be the irreducible components of the cones $C(X,0)$ and $C(Y,0)$ respectively . If there exists a bi-Lipschitz homeomorphism $h : (\mathbb{C}^n, X, 0) \to (\mathbb{C}^n, Y, 0)$, then $r = s$ and, up to a re-ordering of index, $Y_j = dh(X_j)$ and $k_X(X_j) = k_Y(Y_j)$, $\forall\ j$.

Definition of multiplicity

Definition of multiplicity

Here an analytic set of $\mathbb{C}^n$ is an analytic set with pure dimension.

Definition of multiplicity

Here an analytic set of $\mathbb{C}^n$ is an analytic set with pure dimension.

Remark

Let X be an analytic set in $\mathbb{C}^n$ with $d = \dim X$ and $0 \in X$. Then, $\#\pi^{-1}(t) \cap (X \cap U)$ is constant for $\pi : \mathbb{C}^n \to \mathbb{C}^d$ being a generic linear projection and t generic close to $0 \in \mathbb{C}^d$, where U is a neighborhood of $O \in \mathbb{C}^n$ sufficiently small.

Definition of multiplicity

Here an analytic set of $\mathbb{C}^n$ is an analytic set with pure dimension.

Remark

Let X be an analytic set in $\mathbb{C}^n$ with $d = \dim X$ and $0 \in X$. Then, $\#\pi^{-1}(t) \cap (X \cap U)$ is constant for $\pi : \mathbb{C}^n \to \mathbb{C}^d$ being a generic linear projection and t generic close to $0 \in \mathbb{C}^d$, where U is a neighborhood of $O \in \mathbb{C}^n$ sufficiently small.

Definition

In this case, we define the **multiplicity of X at** 0 to be $m(X, 0) = \#\pi^{-1}(t) \cap (X \cap U)$ for $t \in \pi(U)$ generic.

Multiplicity and smoothness

Multiplicity and smoothness

We have a way to decide if a complex analytic set is smooth.

Multiplicity and smoothness

We have a way to decide if a complex analytic set is smooth.

Remark

$(X,0)$ is smooth iff $m(X,0)=1$.

Lelong's numbers and multiplicity

Lelong's numbers and multiplicity

We have a relation between the Lelong's numbers and the multiplicity.

Lelong's numbers and multiplicity

We have a relation between the Lelong's numbers and the multiplicity.

Remark

Let $C(X,0) = X_1 \cup \ldots \cup X_r$ be the decomposition in irreducible components of $C(X,0)$, then

$$m(X,0) = \sum_{j=1}^{r} k_X(X_j) m(X_j,0).$$

Transversal Milnor numbers

Transversal Milnor numbers

Proposition/definition

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a reduced analytic function-germ with $\dim Sing(f) = 1$ and $Sing(f) = C_1 \cup \ldots \cup C_r$. We denote by $\mu'_j(f)$ the Milnor number of f, restricted to a generic hyperplane slice, at a point $p \in C_j \setminus \{0\}$ close to 0. We call the sum $\mu'(f) := \sum_{j=1}^{r} \mu'_j(f)$ the **Transversal Milnor number of** f.

Transversal Milnor numbers

Proposition/definition

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a reduced analytic function-germ with $\dim Sing(f) = 1$ and $Sing(f) = C_1 \cup \ldots \cup C_r$. We denote by $\mu'_j(f)$ the Milnor number of f, restricted to a generic hyperplane slice, at a point $p \in C_j \setminus \{0\}$ close to 0. We call the sum $\mu'(f) := \sum\limits_{j=1}^{r} \mu'_j(f)$ the **Transversal Milnor number of** f.

Theorem (Lê (1973))

Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two reduced analytic function-germs and $h : (\mathbb{C}^n, V(f), 0) \to (\mathbb{C}, V(g), 0)$ be a homeomorphism. Then, $\mu'(f) = \mu'(g)$.

Main results

Lipschitz regularity of complex analytic sets

Lipschitz regularity of complex analytic sets

Theorem (_______ (2016))

Let $X \subset \mathbb{C}^n$ be a complex analytic set. If there is a bi-Lipschitz homeomorphism $h : (X, 0) \to (\mathbb{C}^d, 0)$, then $(X, 0)$ is smooth.

Proof

Proof

- $Y_1 = C(\mathbb{C}^d, 0) = \mathbb{C}^d$. Then $k_{\mathbb{C}^d}(\mathbb{C}^d) = 1$ and $m(\mathbb{C}^d, 0) = 1$.

Proof

- $Y_1 = C(\mathbb{C}^d, 0) = \mathbb{C}^d$. Then $k_{\mathbb{C}^d}(\mathbb{C}^d) = 1$ and $m(\mathbb{C}^d, 0) = 1$.
- By Lipschitz invariance of the tangent cones, $Y_1 = dh(X_1)$ with $X_1 = C(X, 0)$. In particular, X_1 is topologically regular.

Proof

• $Y_1 = C(\mathbb{C}^d, 0) = \mathbb{C}^d$. Then $k_{\mathbb{C}^d}(\mathbb{C}^d) = 1$ and $m(\mathbb{C}^d, 0) = 1$.
• By Lipschitz invariance of the tangent cones, $Y_1 = dh(X_1)$ with $X_1 = C(X, 0)$. In particular, X_1 is topologically regular.
• By Prill's Theorem, X_1 is a plane and then $m(X_1, 0) = 1$.

Proof

• $Y_1 = C(\mathbb{C}^d, 0) = \mathbb{C}^d$. Then $k_{\mathbb{C}^d}(\mathbb{C}^d) = 1$ and $m(\mathbb{C}^d, 0) = 1$.
• By Lipschitz invariance of the tangent cones, $Y_1 = dh(X_1)$ with $X_1 = C(X, 0)$. In particular, X_1 is topologically regular.
• By Prill's Theorem, X_1 is a plane and then $m(X_1, 0) = 1$.
• By bi-Lipschitz invariance of Lelong's numbers, we have $k_X(X_1) = 1$, since $k_Y(Y_1) = 1$.

Proof

- $Y_1 = C(\mathbb{C}^d, 0) = \mathbb{C}^d$. Then $k_{\mathbb{C}^d}(\mathbb{C}^d) = 1$ and $m(\mathbb{C}^d, 0) = 1$.
- By Lipschitz invariance of the tangent cones, $Y_1 = dh(X_1)$ with $X_1 = C(X, 0)$. In particular, X_1 is topologically regular.
- By Prill's Theorem, X_1 is a plane and then $m(X_1, 0) = 1$.
- By bi-Lipschitz invariance of Lelong's numbers, we have $k_X(X_1) = 1$, since $k_Y(Y_1) = 1$.
- But $m(X, 0) = \sum k_X(X_j) \cdot m(X_j, 0) = k_X(X_1) \cdot m(X_1, 0) = 1$.

Proof

- $Y_1 = C(\mathbb{C}^d, 0) = \mathbb{C}^d$. Then $k_{\mathbb{C}^d}(\mathbb{C}^d) = 1$ and $m(\mathbb{C}^d, 0) = 1$.
- By Lipschitz invariance of the tangent cones, $Y_1 = dh(X_1)$ with $X_1 = C(X, 0)$. In particular, X_1 is topologically regular.
- By Prill's Theorem, X_1 is a plane and then $m(X_1, 0) = 1$.
- By bi-Lipschitz invariance of Lelong's numbers, we have $k_X(X_1) = 1$, since $k_Y(Y_1) = 1$.
- But $m(X, 0) = \sum k_X(X_j) \cdot m(X_j, 0) = k_X(X_1) \cdot m(X_1, 0) = 1$.
- Therefore $(X, 0)$ is smooth.

Zariski's Conjecture once more

Zariski's Conjecture once more

Conjecture A

Let $X, Y \subset \mathbb{C}^n$ be two complex analytic sets. If there is a bi-Lipschitz homeomorphism $h : (\mathbb{C}^n, X, 0) \to (\mathbb{C}^n, Y, 0)$, then $m(X, 0) = m(Y, 0)$.

Zariski's Conjecture once more

Conjecture A

Let $X, Y \subset \mathbb{C}^n$ be two complex analytic sets. If there is a bi-Lipschitz homeomorphism $h : (\mathbb{C}^n, X, 0) \to (\mathbb{C}^n, Y, 0)$, then $m(X, 0) = m(Y, 0)$.

Conjecture AH

Let $X, Y \subset \mathbb{C}^n$ be two irreducible homogeneous complex algebraic sets. If there is a bi-Lipschitz homeomorphism $h : (\mathbb{C}^n, X, 0) \to (\mathbb{C}^n, Y, 0)$, then $m(X, 0) = m(Y, 0)$.

Reduction of the Zariski's Conjecture

Reduction of the Zariski's Conjecture

Theorem 2. (Fernandes and _______ (2016))

The Conjecture A has a positive answer if, and only if, the Conjecture AH has a positive answer.

The conjecture AH implies the conjecture A

The conjecture AH implies the conjecture A

• Let $h : (\mathbb{C}^n, X, 0) \rightarrow (\mathbb{C}^n, Y, 0)$ be a bi-Lipschitz homeomorphism.

The conjecture AH implies the conjecture A

- Let $h : (\mathbb{C}^n, X, 0) \to (\mathbb{C}^n, Y, 0)$ be a bi-Lipschitz homeomorphism.
- There is a bi-Lipschitz homeomorphism $dh : (\mathbb{C}^n, X_j, 0) \to (\mathbb{C}^n, Y_j, 0)$, $j = 1, ..., r$,

The conjecture AH implies the conjecture A

- Let $h : (\mathbb{C}^n, X, 0) \to (\mathbb{C}^n, Y, 0)$ be a bi-Lipschitz homeomorphism.
- There is a bi-Lipschitz homeomorphism $dh : (\mathbb{C}^n, X_j, 0) \to (\mathbb{C}^n, Y_j, 0)$, $j = 1, ..., r$,
- where $C(X, 0) = X_1 \cup ... \cup X_r$ and $C(Y, 0) = Y_1 \cup ... \cup Y_r$.

The conjecture AH implies the conjecture A

• Let $h : (\mathbb{C}^n, X, 0) \to (\mathbb{C}^n, Y, 0)$ be a bi-Lipschitz homeomorphism.

• There is a bi-Lipschitz homeomorphism $dh : (\mathbb{C}^n, X_j, 0) \to (\mathbb{C}^n, Y_j, 0)$, $j = 1, ..., r$,

• where $C(X, 0) = X_1 \cup ... \cup X_r$ and $C(Y, 0) = Y_1 \cup ... \cup Y_r$.

• If the Conjecture AH has a positive answer, then $m(X_j, 0) = m(Y_j, 0)$, $j = 1, ..., r$.

The conjecture AH implies the conjecture A

• Let $h : (\mathbb{C}^n, X, 0) \to (\mathbb{C}^n, Y, 0)$ be a bi-Lipschitz homeomorphism.
• There is a bi-Lipschitz homeomorphism $dh : (\mathbb{C}^n, X_j, 0) \to (\mathbb{C}^n, Y_j, 0)$, $j = 1, ..., r$,
• where $C(X, 0) = X_1 \cup ... \cup X_r$ and $C(Y, 0) = Y_1 \cup ... \cup Y_r$.
• If the Conjecture AH has a positive answer, then $m(X_j, 0) = m(Y_j, 0)$, $j = 1, ..., r$.
• By bi-Lipschitz invariance of Lelong's numbers, we have $k_X(X_j) = k_Y(Y_j)$.

The conjecture AH implies the conjecture A

• Let $h : (\mathbb{C}^n, X, 0) \to (\mathbb{C}^n, Y, 0)$ be a bi-Lipschitz homeomorphism.
• There is a bi-Lipschitz homeomorphism $dh : (\mathbb{C}^n, X_j, 0) \to (\mathbb{C}^n, Y_j, 0)$, $j = 1, ..., r$,
• where $C(X, 0) = X_1 \cup ... \cup X_r$ and $C(Y, 0) = Y_1 \cup ... \cup Y_r$.
• If the Conjecture AH has a positive answer, then $m(X_j, 0) = m(Y_j, 0)$, $j = 1, ..., r$.
• By bi-Lipschitz invariance of Lelong's numbers, we have $k_X(X_j) = k_Y(Y_j)$.
•

$$\sum k_X(X_j) \cdot m(X_j, 0) = \sum k_Y(Y_j) \cdot m(Y_j, 0)$$

The conjecture AH implies the conjecture A

- Let $h:(\mathbb{C}^n,X,0)\to(\mathbb{C}^n,Y,0)$ be a bi-Lipschitz homeomorphism.
- There is a bi-Lipschitz homeomorphism $dh:(\mathbb{C}^n,X_j,0)\to(\mathbb{C}^n,Y_j,0)$, $j=1,...,r$,
- where $C(X,0)=X_1\cup...\cup X_r$ and $C(Y,0)=Y_1\cup...\cup Y_r$.
- If the Conjecture AH has a positive answer, then $m(X_j,0)=m(Y_j,0)$, $j=1,...,r$.
- By bi-Lipschitz invariance of Lelong's numbers, we have $k_X(X_j)=k_Y(Y_j)$.
- $m(X,0)=\sum k_X(X_j)\cdot m(X_j,0)=\sum k_Y(Y_j)\cdot m(Y_j,0)$

The conjecture AH implies the conjecture A

- Let $h : (\mathbb{C}^n, X, 0) \to (\mathbb{C}^n, Y, 0)$ be a bi-Lipschitz homeomorphism.
- There is a bi-Lipschitz homeomorphism $dh : (\mathbb{C}^n, X_j, 0) \to (\mathbb{C}^n, Y_j, 0)$, $j = 1, ..., r$,
- where $C(X, 0) = X_1 \cup ... \cup X_r$ and $C(Y, 0) = Y_1 \cup ... \cup Y_r$.
- If the Conjecture AH has a positive answer, then $m(X_j, 0) = m(Y_j, 0)$, $j = 1, ..., r$.
- By bi-Lipschitz invariance of Lelong's numbers, we have $k_X(X_j) = k_Y(Y_j)$.
- $m(X, 0) = \sum k_X(X_j) \cdot m(X_j, 0) = \sum k_Y(Y_j) \cdot m(Y_j, 0) = m(Y, 0).$

The conjecture AH implies the conjecture A

- Let $h : (\mathbb{C}^n, X, 0) \to (\mathbb{C}^n, Y, 0)$ be a bi-Lipschitz homeomorphism.
- There is a bi-Lipschitz homeomorphism $dh : (\mathbb{C}^n, X_j, 0) \to (\mathbb{C}^n, Y_j, 0)$, $j = 1, ..., r$,
- where $C(X, 0) = X_1 \cup ... \cup X_r$ and $C(Y, 0) = Y_1 \cup ... \cup Y_r$.
- If the Conjecture AH has a positive answer, then $m(X_j, 0) = m(Y_j, 0)$, $j = 1, ..., r$.
- By bi-Lipschitz invariance of Lelong's numbers, we have $k_X(X_j) = k_Y(Y_j)$.
- $m(X, 0) = \sum k_X(X_j) \cdot m(X_j, 0) = \sum k_Y(Y_j) \cdot m(Y_j, 0) = m(Y, 0).$
- Therefore the Conjecture A has a positive answer.

Invariance of multiplicity

Invariance of multiplicity

Theorem 3.1. (Fernandes and _______ (2016))

Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two reduced analytic function-germs and $h : (\mathbb{C}^n, V(f), 0) \to (\mathbb{C}^n, V(g), 0)$ be a bi-Lipschitz homeomorphism. If each irreducible component of $C(V(f), 0)$ has isolated singularity at 0, then $m(V(f), 0) = m(V(g), 0)$.

Invariance of multiplicity

Theorem 3.1. (Fernandes and _______ (2016))

Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two reduced analytic function-germs and $h : (\mathbb{C}^n, V(f), 0) \to (\mathbb{C}^n, V(g), 0)$ be a bi-Lipschitz homeomorphism. If each irreducible component of $C(V(f), 0)$ has isolated singularity at 0, then $m(V(f), 0) = m(V(g), 0)$.

By Theorem 2, is sufficiently to prove the following result.

Invariance of multiplicity

Theorem 3.1. (Fernandes and _______ (2016))

Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two reduced analytic function-germs and $h : (\mathbb{C}^n, V(f), 0) \to (\mathbb{C}^n, V(g), 0)$ be a bi-Lipschitz homeomorphism. If each irreducible component of $C(V(f), 0)$ has isolated singularity at 0, then $m(V(f), 0) = m(V(g), 0)$.

By Theorem 2, is sufficiently to prove the following result.

Theorem (Fernandes and _______ (2016))

Let $f, g : \mathbb{C}^n \to \mathbb{C}$ be two irreducible homogeneous polynomials and $h : (\mathbb{C}^n, V(f), 0) \to (\mathbb{C}^n, V(g), 0)$ be a homeomorphism. If $V(f)$ has isolated singularity at 0, then $m(V(f), 0) = m(V(g), 0)$.

Proof

Proof

Define $d = m(V(f), 0)$ and $e = m(V(g), 0)$.

Proof

Define $d = m(V(f), 0)$ and $e = m(V(g), 0)$. By Theorem of A'Campo-Lê, we can suppose $e, d > 1$.

Proof

Define $d = m(V(f), 0)$ and $e = m(V(g), 0)$. By Theorem of A'Campo-Lê, we can suppose $e, d > 1$.
If $V(f)$ has isolated singularity, then by Theorem of A'Campo-Lê, $V(f)$ has isolated singularity, as well.

Proof

Define $d = m(V(f), 0)$ and $e = m(V(g), 0)$. By Theorem of A'Campo-Lê, we can suppose $e, d > 1$.
If $V(f)$ has isolated singularity, then by Theorem of A'Campo-Lê, $V(f)$ has isolated singularity, as well. The Theorem follows from

$$(d-1)^n = \mu(f) = \mu(g) = (e-1)^n. \qquad (1)$$

Invariance of multiplicity

Invariance of multiplicity

Theorem 3.2. (Fernandes and _______ (2016))

Let $f, g : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be two reduced analytic function-germs and $h : (\mathbb{C}^3, V(f), 0) \to (\mathbb{C}^3, V(g), 0)$ be a bi-Lipschitz homeomorphism. Then $m(V(f), 0) = m(V(g), 0)$.

Theorem

Let $f, g : \mathbb{C}^3 \to \mathbb{C}$ be irreducible homogeneous polynomials and $h : (\mathbb{C}^3, V(f), 0) \to (\mathbb{C}^3, V(g), 0)$ be a homeomorphism. Then $m(V(f), 0) = m(V(g), 0)$.

Invariance of multiplicity

Theorem 3.2. (Fernandes and _______ (2016))

Let $f, g : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be two reduced analytic function-germs and $h : (\mathbb{C}^3, V(f), 0) \to (\mathbb{C}^3, V(g), 0)$ be a bi-Lipschitz homeomorphism. Then $m(V(f), 0) = m(V(g), 0)$.

By Theorem 2, is sufficiently to prove the following result.

Theorem

Let $f, g : \mathbb{C}^3 \to \mathbb{C}$ be irreducible homogeneous polynomials and $h : (\mathbb{C}^3, V(f), 0) \to (\mathbb{C}^3, V(g), 0)$ be a homeomorphism. Then $m(V(f), 0) = m(V(g), 0)$.

Proof

Proof

Define $d = m(V(f), 0)$ and $e = m(V(g), 0)$.

Proof

Define $d = m(V(f), 0)$ and $e = m(V(g), 0)$. By the last Therem, we can suppose that $\dim Sing(f) = \dim Sing(g) = 1$ and by Theorem of A'Campo-Lê, we can suppose $e, d > 1$.

Proof

Define $d = m(V(f), 0)$ and $e = m(V(g), 0)$. By the last Therem, we can suppose that $\dim Sing(f) = \dim Sing(g) = 1$ and by Theorem of A'Campo-Lê, we can suppose $e, d > 1$.

Proof

Define $d = m(V(f), 0)$ and $e = m(V(g), 0)$. By the last Therem, we can suppose that $\dim Sing(f) = \dim Sing(g) = 1$ and by Theorem of A'Campo-Lê, we can suppose $e, d > 1$.

Proof

Define $d = m(V(f), 0)$ and $e = m(V(g), 0)$. By the last Therem, we can suppose that $\dim Sing(f) = \dim Sing(g) = 1$ and by Theorem of A'Campo-Lê, we can suppose $e, d > 1$. If $n = 3$, then we have the Lê-Iomdin formulas

Proof

Define $d = m(V(f), 0)$ and $e = m(V(g), 0)$. By the last Therem, we can suppose that $\dim Sing(f) = \dim Sing(g) = 1$ and by Theorem of A'Campo-Lê, we can suppose $e, d > 1$. If $n = 3$, then we have the Lê-Iomdin formulas

$$(d-1)^3 = \chi(F_f) - 1 + d\mu'(f) \tag{2}$$

$$(e-1)^3 = \chi(F_g) - 1 + e\mu'(g) \tag{3}$$

Proof

Define $d = m(V(f), 0)$ and $e = m(V(g), 0)$. By the last Therem, we can suppose that $\dim Sing(f) = \dim Sing(g) = 1$ and by Theorem of A'Campo-Lê, we can suppose $e, d > 1$. If $n = 3$, then we have the Lê-Iomdin formulas

$$(d-1)^3 = \chi(F_f) - 1 + d\mu'(f) \tag{2}$$

$$(e-1)^3 = \chi(F_g) - 1 + e\mu'(g) \tag{3}$$

and we have $\mu' = \mu'(f) = \mu'(g)$.

Proof

Proof

- If $\chi(F_f) = 0$ (and thus $\chi(F_g) = 0$), then d and e are roots of

Proof

- If $\chi(F_f) = 0$ (and thus $\chi(F_g) = 0$), then d and e are roots of

$$x^2 - 3x + 3 - \mu' = 0 \tag{4}$$

Proof

- If $\chi(F_f) = 0$ (and thus $\chi(F_g) = 0$), then d and e are roots of

$$x^2 - 3x + 3 - \mu' = 0 \tag{4}$$

Proof

- If $\chi(F_f) = 0$ (and thus $\chi(F_g) = 0$), then d and e are roots of

$$x^2 - 3x + 3 - \mu' = 0 \tag{4}$$

But the equation (4) has only one solution greater than 1.

Proof

- If $\chi(F_f) = 0$ (and thus $\chi(F_g) = 0$), then d and e are roots of

$$x^2 - 3x + 3 - \mu' = 0 \tag{4}$$

But the equation (4) has only one solution greater than 1.
Thus $d = e$.

Proof

Proof

- If $\chi(F_f) \neq 0$ (and thus $\chi(F_g) \neq 0$), then

Proof

- If $\chi(F_f) \neq 0$ (and thus $\chi(F_g) \neq 0$), then

We can take the monodromy homeomorphism as $h_f(x) = e^{\frac{2\pi i}{d}} x$.

Proof

- If $\chi(F_f) \neq 0$ (and thus $\chi(F_g) \neq 0$), then

We can take the monodromy homeomorphism as $h_f(x) = e^{\frac{2\pi i}{d}} x$. Using the Topological Cylindric Structure at infinity, we have that F_f has the same homotopy type of $F = \{x \in \mathbb{C}^n; \|x\| \leq R\} \cap F_f$, for R large enough.

Proof

• If $\chi(F_f) \neq 0$ (and thus $\chi(F_g) \neq 0$), then
We can take the monodromy homeomorphism as $h_f(x) = e^{\frac{2\pi i}{d}} x$.
Using the Topological Cylindric Structure at infinity, we have that F_f has the same homotopy type of $F = \{x \in \mathbb{C}^n; \|x\| \leq R\} \cap F_f$, for R large enough. h_f and $\bar{h}_f := h_f|_F : F \to F$ has the same homotopy type.

Proof

- If $\chi(F_f) \neq 0$ (and thus $\chi(F_g) \neq 0$), then

We can take the monodromy homeomorphism as $h_f(x) = e^{\frac{2\pi i}{d}} x$. Using the Topological Cylindric Structure at infinity, we have that F_f has the same homotopy type of $F = \{x \in \mathbb{C}^n; \|x\| \leq R\} \cap F_f$, for R large enough. h_f and $\bar{h}_f := h_f|_F : F \to F$ has the same homotopy type.

If $0 < k < d$, then $\bar{h}_f^{\,k}$ does not have fixed point.

Proof

- If $\chi(F_f) \neq 0$ (and thus $\chi(F_g) \neq 0$), then

We can take the monodromy homeomorphism as $h_f(x) = e^{\frac{2\pi i}{d}} x$.
Using the Topological Cylindric Structure at infinity, we have that F_f has the same homotopy type of $F = \{x \in \mathbb{C}^n; \|x\| \leq R\} \cap F_f$, for R large enough. h_f and $\bar{h}_f := h_f|_F : F \to F$ has the same homotopy type.
If $0 < k < d$, then $\bar{h}_f^{\,k}$ does not have fixed point.
By Lefschetz's fixed point Theorem, $\Lambda(h_f^k) = \Lambda(\bar{h}_f^{\,k}) = 0$.

Proof

Proof

- As $\Lambda(\bar{h_f}^d) = \Lambda(h_f^d) = \Lambda(id_{F_f}) = \chi(F_f) \neq 0$, then

Proof

- As $\Lambda(\bar{h_f}^d) = \Lambda(h_f^d) = \Lambda(id_{F_f}) = \chi(F_f) \neq 0$, then

$$d = min\{k \in \mathbb{N} \setminus \{0\};\ \Lambda(h_f^k) \neq 0\}$$

Proof

- As $\Lambda(\bar{h_f}^d) = \Lambda(h_f^d) = \Lambda(id_{F_f}) = \chi(F_f) \neq 0$, then

$$d = min\{k \in \mathbb{N} \setminus \{0\};\ \Lambda(h_f^k) \neq 0\}$$

and likewise

Proof

- As $\Lambda(\bar{h}_f{}^d) = \Lambda(h_f^d) = \Lambda(id_{F_f}) = \chi(F_f) \neq 0$, then

$$d = min\{k \in \mathbb{N} \setminus \{0\};\ \Lambda(h_f^k) \neq 0\}$$

and likewise

$$e = min\{k \in \mathbb{N} \setminus \{0\};\ \Lambda(h_g^k) \neq 0\}.$$

Proof

- As $\Lambda(\bar{h_f}^{d}) = \Lambda(h_f^d) = \Lambda(id_{F_f}) = \chi(F_f) \neq 0$, then

$$d = min\{k \in \mathbb{N} \setminus \{0\}; \Lambda(h_f^k) \neq 0\}$$

and likewise

$$e = min\{k \in \mathbb{N} \setminus \{0\}; \Lambda(h_g^k) \neq 0\}.$$

Using the homotopy invariance of Lefschetz's numbers, we have

Proof

- As $\Lambda(\bar{h_f}^d) = \Lambda(h_f^d) = \Lambda(id_{F_f}) = \chi(F_f) \neq 0$, then

$$d = min\{k \in \mathbb{N} \setminus \{0\};\ \Lambda(h_f^k) \neq 0\}$$

and likewise

$$e = min\{k \in \mathbb{N} \setminus \{0\};\ \Lambda(h_g^k) \neq 0\}.$$

Using the homotopy invariance of Lefschetz's numbers, we have

$$d = e.$$

Thanks!

A'Campo, N.
Le nombre de Lefschetz d'une monodromie.(French)
Nederl. Akad. Wetensch. Proc. Ser. A 76 = Indag. Math., vol. 35, pp. 113–118, 1973.

BIRBRAIR, L.; FERNANDES, A.; LÊ D. T. and SAMPAIO, J. E.
Lipschitz regular complex algebraic sets are smooth,
Proceedings of the American Mathematical Society, v. 144, pp. 983–987, 2016.

BIRBRAIR, L.; FERNANDES, A. and GRANDJEAN, V.
Thin-thick decomposition for real definable isolated singularities,
preprint 2014, http://arxiv.org/pdf/1505.07309v1.pdf.

BOBADILLA, J. F.
Answers to some equisingularity questions.

Invent. math., vol. 161, pp. 657–675, 2005.

CHIRKA, E. M.
Complex analytic sets.
Kluwer Academic Publishers, 1989.

COMTE, Georges.
Multiplicity of complex analytic sets and bi-Lipschitz maps.
Real analytic and algebraic singularities (Nagoya/Sapporo/Hachioji, 1996) Pitman Res. Notes Math. Ser., vol. 381, pp.182–188, 1998.

DIMCA, Alexandru
Singularities and topology of hypersurfaces.
New york: Springer-Verlag, 1992.

DRAPER, R. N.
Intersection Theory in Analytic Geometry.
Mathematische Annalen, vol. 180, pp. 175–204, 1969.

EPHRAIM, R.
C^1 preservation of multiplicity.
Duke Math., vol. 43, pp. 797–803, 1976.

EYRAL, C.
Zariski's multiplicity question and aligned singularities.
Comptes Rendus Mathematique, vol. 342, Issue 3, pp. 183–186, 2006.

EYRAL, C.
Zariski's multiplicity questions - A survey.
New Zealand Journal of Mathematics, vol. 36, pp. 253–276, 2007.

GAU, Y.-N. and LIPMAN, J.
Differential invariance of multiplicity on analytic varieties.
Inventiones mathematicae, vol. 73 (2), pp. 165–188, 1983.

GREUEL, Gert-Martin
Constant Milnor number implies constant multiplicity for quasihomogeneous singularities..
Manuscripta Math. 56 (2), pp. 159–166, 1986.

IOMDIN, I. N.
Complex surfaces with a one-dimensional set of singularities.
Siberian Mathematical Journal, vol. 15 (5), pp. 748–762, 1974. (Translated from Sibirskii Matematicheskii Zhurnal, Vol. 15, No. 5, pp. 1061–1082, September-October, 1974).

KARRAS, U.
Equimultiplicity of deformations of constant Milnor number.
Proceedings of the conference on algebraic geometry (Berlin, 1985), pp. 186–209, Teubner-Texte Math., 92, Teubner, Leipzig, 1986.

KURDYKA, K. and RABY, G.

Densité des ensembles sous-analytiques.
Ann. Inst. Fourier (Grenoble), vol. 39 (3), pp. 753–771, 1989.

LÊ D. T.
Topologie des singularités des hypersurfaces complexes.
(French) Singularités à Cargèse, Asterisque, vol. 7 and 8, pp. 171–182, 1973.

LÊ D. T.
Calcul du nombre de cycles évanouissants d'une hypersurface complexe.
(French) Ann. Inst. Fourier (Grenoble), vol. 23 (4), pp. 261–270, 1973.

LÊ D. T.
Ensembles analytiques complexes avec lieu singulier de dimension un.

(D'Apres I. N. Iomdine), Seminar on Singularities (Paris, 1976/1977), Publ. Math. Univ. Paris VII 7, pp. 87–95, 1980.

MASSEY, David B.
Lê cycles and hypersurfaces singularities.
Lecture Notes Math. 1615, Berlin: Springer-Verlag, 1995.

MASSEY, David B.
Lê's Work on Hypersurface Singularities.
preprint 2007, `http://www.matcuer.unam.mx/~singularities/Notas/Massey1.pdf`.

MILNOR, J.
Singular points of complex hypersurfaces.
Princeton: Princeton University Press, 1968.

NEUMANN, W. and PICHON, A.
Lipschitz geometry of complex surfaces: analytic invariants and equisingularity.

preprint 2016, http://arxiv.org/pdf/1211.4897v3.pdf

PRILL, David.
Topologically trivial deformations of isolated quasihomogeneous hypersurface singularities are equimultiple.
Proc. Amer. Math. Soc., vol. 101 (2), pp. 260–262, 1987.

PRILL, David.
Cones in complex affine space are topologically singular.
Proc. Amer. Math. Soc., vol. 18, pp. 178–182, 1967.

RANDELL, R.
On the topology of non-isolated singularities.
Geometric Topology. Academic Press, New York-London, pp. 445–473, 1979.

SAEKI, O..
Topological invariance of weights for weighted homogeneous isolated singularities in $\mathbb{C}^3$.

Proc. Amer. Math. Soc., vol. 103 (3), pp. 905–909, 1988.

SAMPAIO, J. Edson
Bi-Lipschitz homeomorphic subanalytic sets have bi-Lipschitz homeomorphic tangent cones.
Selecta Math. (N.S.), v. online (2015), 1–7 (DOI: 10.1007/s00029-015-0195-9).

TROTMAN, D.
Multiplicity is a C^1 invariant.
University Paris 11 (Orsay), Preprint, 1977.

VALETTE, Guillaume.
Multiplicity mod 2 as a metric invariant.
Discrete Comput. Geom., 43: 663–679, 2010.

YAU, Stephen S.-T.

Topological types and multiplicities of isolated quasi-homogeneous surface singularities.
Bulletin (new series) of the American Mathematical Society, vol. 19 (2), pp. 447–454, 1988.

WHITNEY, H.
Complex Analytic Varieties.
California: Addison-Wesley publishing company, 1972.

ZARISKI, O.
Some open questions in the theory of singularities.
Bull. of the Amer. Math. Soc., vol. 77 (4), pp. 481–491, 1971.