Closed orbits, flags, and integrability for singularities of complex vector fields in dimension three

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14th International Workshop on Real and Complex Singularities - São Carlos - ICMC/USP

July 28, 2016

Holomorphic first integral

Definition (1)

We say that a germ of holomorphic foliation $\mathcal{F}(X)$ *has a holomorphic first integral*, if there is a germ of holomorphic map $F\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^{n-1}, 0)$ such that:

(a) F is a submersion almost everywhere, i.e., if we write $\mathrm{F} = (\mathrm{f}_1, \cdots, \mathrm{f}_{n-1})$ in coordinate functions, then the $(n-1)$-form $\mathrm{df}_1 \wedge \cdots \wedge \mathrm{df}_{n-1}$ is non-identically zero, equivalently, it has maximal rank except for a proper analytic subset;

(b) The leaves of $\mathcal{F}(X)$ are contained in level curves of F.

$\mathcal{F}(X)$-invariant meromorphic functions

Definition

Further, a germ f of a meromorphic function at the origin $0 \in \mathbb{C}^n$ is called $\mathcal{F}(X)$-*invariant* if the leaves of $\mathcal{F}(X)$ are contained in the level sets of f. This can be precisely stated in terms of representatives for $\mathcal{F}(X)$ and f, but can also be written as $i_X(df) = X(f) \equiv 0$.

generic germs

Definition

We say that $\mathcal{F}(X)$ is *non-degenerate generic* if $DX(0)$ is non-singular, diagonalizable and, after some suitable change of coordinates, X leaves invariant the coordinate planes.

First integrals and resonance

- A generic vector field $X \in (\mathfrak{X}(\mathbb{C}^3, 0))$ has a holomorphic first integral $F = (f_1, f_2)$ if and only if $i_X df_j = 0$, $j = 1, 2$, and f_1, f_2 are transversal off the singular set of X.
- Consider a vector field

$$X(x) = \sum_{j=1}^{3} \lambda_j x_j (1 + a_j(x)) \frac{\partial}{\partial x_j}$$

with $a_j \in \mathcal{M}_3$, then any $\mathcal{F}$-invariant holomorphic function must be of the form $f(x) = \sum_{|N| \geq p} a_N x^N$, $a_N \in \mathbb{C}$, $p \geq 2$ and $N \in \mathbb{N}^3 - C_3$ with $C_3 := \{(n_1, n_2, n_3) \in \mathbb{N}^3 : n_1 n_2 n_3 = 0\}$.

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First integrals and resonance

- Since $\frac{\partial f}{\partial x_j} = \sum_{|N|\geq p} n_j a_N x^N x_j^{-1}$, then

$$J^p(df(X)) = \frac{\partial f(x)}{\partial x_1}\cdot(\lambda_1 x_1) + \frac{\partial f(x)}{\partial x_2}\cdot(\lambda_2 x_2) + \frac{\partial f(x)}{\partial x_3}\cdot(\lambda_3 x_3)$$
$$= \sum_{|N|=p} (\lambda_1 n_1 + \lambda_2 n_2 + \lambda_3 n_3) a_N x^N.$$

- From $i_X df = 0$ we obtain

$$0 = (\lambda_1 n_1 + \lambda_2 n_2 + \lambda_3 n_3) a_N \text{ for all } |N| = p,\ N \in \mathbb{N}^3 - C_3. \qquad (1)$$

- Thus in the absence of a resonance of the form

$$\lambda_1 n_1 + \lambda_2 n_2 + \lambda_3 n_3 = 0, \qquad (2)$$

there will be no first integral.

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Lemma

Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}^*$, *and let* $(n_1, n_2, n_3), (m_1, m_2, m_3) \in \mathbb{N}^3 - C_3$ *be linearly independent and satisfying* (2) *above. Then there are* $m, n, k \in \mathbb{Z}$ *and* $\lambda \in \mathbb{C}^*$ *such that*

$$(\lambda_1, \lambda_2, \lambda_3) = \lambda(m, n, k)$$

and $m \cdot n \cdot k < 0$.

Proposition

Suppose that $X \in \text{Gen}(\mathfrak{X}(\mathbb{C}^3, 0))$ *is generic and has a holomorphic first integral, then* $\mathcal{F}_X$ *can be given in local coordinates by a vector field of the form*

$$X(x) = mx_1(1 + a_1(x))\frac{\partial}{\partial x_1} + nx_2(1 + a_2(x))\frac{\partial}{\partial x_2} - kx_3(1 + a_3(x))\frac{\partial}{\partial x_3}$$

where $m, n, k \in \mathbb{Z}_+$ *and* $a_1, a_2, a_3 \in \mathcal{M}_3$.

Sketch of proof

- Suppose that $J^1(X) = \lambda_1 x_1 \frac{\partial}{\partial x_1} + \lambda_2 x_2 \frac{\partial}{\partial x_2} + \lambda_3 x_3 \frac{\partial}{\partial x_3}$, then Lemma 4 assures that its enough to prove that there is a pair of linearly independent vectors $M, N \in \mathbb{N}^3 - C_3$ satisfying (2).
- Suppose $F = (f, g)$ is a first integral for X, with $f(x) = \sum_{|N| \geq p} a_N x^N$ and $g(x) = \sum_{|N| \geq q} b_N x^N$. From (2) we have $0 = (\lambda_1 n_1 + \lambda_2 n_2 + \lambda_3 n_3) a_N$ for all $|N| = p$. If there are two distinct $a_N, a_{N'} \neq 0$, then N and N' satisfy the desired condition.
- Reasoning in the same manner for g we just have to consider the case $f(x) = a_P x^P + \sum_{|N| \geq p+1} a_N x^N$ and $g(x) = b_P x^P + \sum_{|N| \geq p+1} b_N x^N$ with $|P| = p$, and $a_P, b_P \neq 0$.

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- Now let $f_1 := \frac{1}{a_P} f - \frac{1}{b_P} g$, then it can be written in the form $f_1(x) = h_1(x^P) + \sum_{|N|=q, N \notin \langle P \rangle} c_N x^N + \cdots$, where $h_1 \in \mathcal{M}$ is a polynomial such that $\tau_1 := \deg(h_1) < q$, where $q = |N|$ is the least natural number such that there exists $c_N \neq 0$ form some $N \notin \langle P \rangle$, where $\langle P \rangle$ denotes the ideal in $\mathbb{N}^n$ generated by the coordinates of P (notice that such q exists, since f and g are transversal off the origin).
- Pick inductively $f_k := f_{k-1} - h_{k-1}^{(\tau_{k-1})}(0) \left(\frac{1}{b_P} g\right)^{\tau_{k-1}}$, where $\tau_k := \deg(f_k)$, then after repeating this process a finite number of steps we have $k_0 \in \mathbb{Z}_+$ such that $f_{k_0}(x) = \sum_{|N|=q, N \notin \langle P \rangle} c_N x^N + \cdots$
- Since the set of $\mathcal{F}_X$-invariant holomorphic functions is a sub-ring of $\mathcal{O}_3$, then f_{k_0} is an $\mathcal{F}_X$-invariant holomorphic function; in particular it satisfies (1). Therefore, it is enough to pick $R \notin \langle P \rangle$ such that $|R| = q$ and $c_R \neq 0$.

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Definition

Let $X \in \mathrm{Gen}(\mathfrak{X}(\mathbb{C}^3, 0))$. We say that X satisfies condition $(\star)$ if there is a real line $L \subset C$ through the origin containing the eigenvalues of X such that one of the connected components $L \setminus \{0\}$ contains a single eigenvalue $\lambda(X)$ of X. In other words, not all the eigenvalues of X belong to the same connected component of $L \setminus \{0\}$.

Lemma

Let $N, M \in \mathbb{N}^3 - C_3$ be two vectors satisfying (2), and let $f(x) = x^N$, $g(x) = x^M$. Then $\mathrm{Sat}(df = 0)$ is transversal to $\mathrm{Sat}(dg = 0)$ if and only if N and M are linearly independent.

An algebraic characterization of integrable linear vector fields is given by the following result.

Lemma

Any linear vector field of the form $X(x) = mx_1\frac{\partial}{\partial x_1} + nx_2\frac{\partial}{\partial x_2} - kx_3\frac{\partial}{\partial x_3}$, where $(m,n,k) \in \mathbb{Z}^3_+$, has a holomorphic first integral of the form $F(x) = (x^N, x^M)$, where $N, M \in \mathbb{N}^3 - C_3$.

Proof.

From Lemma 7 and the calculations made in order to obtain (1), one can easily check that this is just a matter of finding two linearly independent solutions in $\mathbb{N}^3 - C_3$ for the homogeneous equation $mx + ny - kz = 0$. Therefore, we just have to pick $x_j := k\widetilde{x}_j$ and $y_j := k\widetilde{y}_j$, $j = 1, 2$, where $(\widetilde{x}_1, \widetilde{y}_1), (\widetilde{x}_2, \widetilde{y}_2) \in \mathbb{N}^2$ are linearly independent. □

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Let $X \in \text{Gen}(\mathfrak{X}(\mathbb{C}^3, 0))$ be given by

$$X(x) = -\frac{m_1 x_1}{k}(1 + a_1(x))\frac{\partial}{\partial x_1} - \frac{m_2 x_2}{k}(1 + a_2(x))\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}$$

where $m_1, m_2, k \in \mathbb{Z}_+$, $S := (x_1 = x_2 = 0)$ and $\Sigma := (x_3 = 1)$, and $\langle h \rangle = \text{Hol}(\mathcal{F}_X, S, \Sigma)$. We conclude that h is resonant.

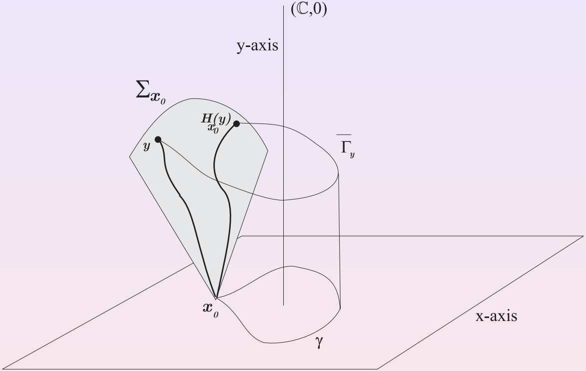

Figure: The lifting of γ along the leaves of $\mathcal{F}$.

Now consider the closed loop $\gamma : [0,1] \longrightarrow S$ given by $\gamma(t) = (0, 0, e^{2\pi \mathbf{i} t})$ and let $\overline{\Gamma}_{(x_1,x_2)}(t) = (\Gamma_1(t, x_1, x_2), \Gamma_2(t, x_1, x_2), \gamma(t))$ be its lifting along the leaves of $\mathcal{F}_X$ starting at $(x_1, x_2, 1) \in \Sigma$. In particular, the map $h \in \mathrm{Diff}(\mathbb{C}^2, 0)$ given by $\overline{\Gamma}_{(x_1,x_2)}(1) = (h(x_1, x_2), 1)$ is a generator of $(\mathcal{F}_X, S, \Sigma)$. Since $\overline{\Gamma}_{(x_1,x_2)}(t)$ belongs to a leaf of $\mathcal{F}_X$, then $\frac{\partial}{\partial t}\overline{\Gamma}_{(x_1,x_2)}(t) = \alpha X(\Gamma_1(t, x_1, x_2), \Gamma_2(t, x_1, x_2), \gamma(t))$. From this vector equation we obtain that $\gamma'(t) = \alpha\gamma(t)$, and thus $\alpha = 2\pi\mathbf{i}$. Furthermore

$$\frac{\partial}{\partial t}\Gamma_n = -\frac{2m_j\pi\mathbf{i}}{k}\Gamma_j(1 + a_j(\Gamma_1, \Gamma_2, \gamma)), \qquad j = 1, 2;$$

If we let $\Gamma_n(t,x_1,x_2)=\sum_{i+j\geq 1} c^n_{i,j}(t)x_1^i x_2^j$ and consider the first jet in the variables (x_1,x_2) of the above equations, then

$$(c^n_{i,j})'(t)=-\frac{2m_j\pi\mathbf{i}}{k}c^n_{i,j}(t),\quad i,j,n-1\in\{0,1\}. \tag{3}$$

Recall that $\Gamma_n(0,x_1,x_2)=x_n$ thus

$$\begin{pmatrix} c^1_{1,0}(0) & c^1_{0,1}(t)\\ c^2_{1,0}(t) & c^2_{0,1}(t)\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

Thus $c^2_{1,0}(t)=c^1_{0,1}(t)=0$, $c^1_{1,0}(t)=\exp(-\frac{2m_1\pi\mathbf{i}}{k}t)$ and $c^2_{0,1}(t)=\exp(-\frac{2m_2\pi\mathbf{i}}{k}t)$ are the solutions for (3). In particular

$$h'(0,0)=\begin{pmatrix} c^1_{1,0}(1) & c^1_{0,1}(1)\\ c^2_{1,0}(t) & c^2_{0,1}(t)\end{pmatrix}=\begin{pmatrix}\exp(-\frac{2m_1\pi\mathbf{i}}{k}) & 0\\ 0 & \exp(-\frac{2m_2\pi\mathbf{i}}{k})\end{pmatrix} \tag{4}$$

Therefore, h is resonant. Notice that the same above computation shows that if $X \in \text{Gen}(\mathfrak{X}(\mathbb{C}^3, 0))$ is given by

$$X(x) = -\lambda_1 x_1(1 + a_1(x))\frac{\partial}{\partial x_1} - \lambda_2 x_2(1 + a_2(x))\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3},$$

thus the holonomy map $h(x_1, x_2)$ has linear part given by

$$h'(0,0) = \begin{pmatrix} \exp(-2\pi \mathbf{i}\lambda_1) & 0 \\ 0 & \exp(-2\pi \mathbf{i}\lambda_2) \end{pmatrix} \tag{5}$$

In particular, we conclude that if h has finite orbits, then necessarily $\lambda_1, \lambda_2 \in \mathbb{Q}$ (indeed, this is quite well-known for one-dimensional germs of diffeomorphisms and one just to consider the restriction of h to the coordinates axes x_1 and x_2 to use this case).

Orbits

Let $G \in \mathrm{Diff}(\mathbb{C}^2, 0)$ and V a neighborhood of the origin where a representative (also denoted by G) of the germ G is defined. Then we denote by

$$\mathcal{O}_V^+(G, x) = \left\{ G^{\circ(n)}(x) : \ G^{\circ(j)}(x) \in V, j = 0, \dots, n \right\}$$

the so called *positive semiorbit* of $x \in V$ by G. Analogously, the *negative semiorbit* of $x \in V$ by G is the set $\mathcal{O}_V^-(G, x) := \mathcal{O}_V^+(G^{-1}, x)$. The *orbit* of $x \in V$ by G is the set $\mathcal{O}_V(G, x) = \mathcal{O}_V^+(G, x) \cup \mathcal{O}_V^-(G, x)$. The cardinality of $\mathcal{O}_V(G, x)$ is denoted by $|\mathcal{O}_V(G, x)|$.

Orbits

Theorem (Brochero Martínez [4])

Let $G \in \mathrm{Diff}(\mathbb{C}^2, 0)$, then the group generated by G is finite if and only if there exists a neighborhood V of the origin such that $|\mathcal{O}_V(G, x)| < \infty$ for all $x \in V$ and G preserves infinitely many analytic invariant curves at 0.

Using the same arguments as in the one-dimensional case (cf. [10], Proposition 1.1, p. 475-476), one can prove that a finite abelian (e.g., cyclic) subgroup of $\mathrm{Diff}(\mathbb{C}^n, 0)$ is always periodic, i.e., it is generated by a periodic (and linearizable) element. Contrasting with the one dimensional case, in greater dimensions the finiteness of the orbits in not enough to ensure the periodicity of the group (cf. [10], Theorem 2, p. 477).

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Orbits

Example

Consider the map $G(x, y) = (x + y^2, y)$. The orbits of G are confined in the level sets of $f(x, y) = y$ and are clearly finite. Notice that $\#\mathcal{O}_V(G, (x, y)) \to \infty$ as $y \to 0$, thus G is not periodic nor linearizable. Furthermore, the orbits $\mathcal{O}_V(G, (x, y))$ are far from being stable, since in each line $(y = c)$ the map G acts as a translation.

Proposition

Let $f, g \in \mathcal{O}_2$ be generically transverse germs and $G \in \text{Diff}(\mathbb{C}^2, 0)$ be a complex map germ having finite orbits and preserving the level sets of both f and g. Then G is periodic.

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Let $f, g \in \mathcal{O}_2$ be generically transverse germs and $G \in \mathrm{Diff}(\mathbb{C}^2, 0)$ be a complex map germ having finite orbits and preserving the level sets of both f and g. Then G is periodic.

Idea of the proof

Since f and g are generically transverse, then one can find a pure meromorphic function $h_o = f_o/g_o$ whose level sets are preserved by G. Hence the infinitely many curves $f_o(x,y) - c \cdot g_o(x,y) = 0$ with $c \in (\mathbb{C}, 0)$ pass through the origin and are invariant by G. Thus Theorem 9 ensures that G is periodic.

Sketch of proof

- Now let us construct h_o. If f/g is already pure meromorphic, then it is enough to pick $h_o := f/g$.
- Otherwise one has $f = h \cdot g^k$, where $k \in \mathbb{Z}_+$, and h is a germ of holomorphic function not divisible by g. Clearly, h is G-invariant, thus if it has an irreducible component distinct from the irreducible components of g, then h/g must be a G-invariant pure meromorphic function.
- Suppose that the decomposition in irreducible components of g and h are of the form $g = g_1^{p_1} \cdots g_n^{p_n}$ and $h = g_1^{q_1} \cdots g_n^{q_n}$. Since h is not divisible by g, then there must be $j_0 \in \{1, \cdots, n\}$ such that $q_{j_0} < p_{j_0}$. If there is also $j_1 \in \{1, \cdots, n\}$ such that $q_{j_1} > p_{j_1}$, then h/g is a pure meromorphic G-invariant function.

Sketch of proof

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Sketch of proof

- From now on we suppose that $q_j \leq p_j$ for all $j = 1, \ldots, n$ with at least one $j_0 \in \{1, \cdots, n\}$ such that $q_{j_0} < p_{j_0}$. If there is $j_1 \in \{1, \cdots, n\}$ such that $q_{j_1} = p_{j_1}$, then after reordering the indexes (if necessary) we may suppose that: (i) $q_i < p_i$ for all $i = 1, \ldots, n_0$; (ii) $q_i = p_i$ for all $i = n_0 + 1, \cdots, n$; for some $n_0 \in \{1, \cdots, n-1\}$. Then $\overline{h} := g/h = g_1^{p_1 - q_1} \cdots g_{n_0}^{p_{n_0} - q_{n_0}}$ is a G-invariant germ of a holomorphic function. Now, let $s_1 := [p_1/(p_1 - q_1)] + 1$ (where $[x]$ denotes the integer part of $x \in \mathbb{R}$), then a straightforward calculation shows that $g/\overline{h}^{s_1}$ is a pure meromorphic function.

Sketch of proof

- Hereafter we suppose that $q_j < p_j$ for all $j = 1, \ldots, n$. Recall that the Euclid's algorithm of a pair of positive integers (p, q), $p > q$, is the sequence of pairs of positive integers $\{(p_j, q_j)\}_{j=1}^{n+1}$ given by: (1) $(p_{j+1}, q_{j+1}) := (p, q)$; (2) $p_j = q_j \cdot r_j + s_j$, where $r_j := [p/q]$ and $s_j < q_j$; (3) $(p_{j+1}, q_{j+1}) := (q_j, r_j)$; and (4) $s_n > 0$ and $s_{n+1} = 0$. This is called the Euclid's sequence of the pair (p, q).

Sketch of proof

- For simplicity, suppose that g and h have only two irreducible components, say $g = f^p(\overline{f})^{\overline{p}}$ and $h = f^q(\overline{f})^{\overline{q}}$, and let $\{(p_j, q_j)\}_{j=1}^{n+1}$ and $\{(\overline{p}_j, \overline{q}_j)\}_{j=1}^{n+1}$ be the Euclid's sequence of (p, q) and $(\overline{p}, \overline{q})$, respectively. If $r_1 = [p_1/q_1] < [\overline{p}_1/\overline{q}_1] = \overline{r}_1$, then $p_1 - (r_1 + 1)q_1 < 0$ and $\overline{p}_1 - (\overline{r}_1 + 1)\overline{q}_1 \geq 0$. If $\overline{p}_1 - (\overline{r}_1 + 1)q_1 \neq 0$, then g/h^{r_1+1} is a G-invariant germ of a pure meromorphic function, otherwise $g/h^{r_1+1} = 1/f^{(r_1+1)q_1-p_1}$ and $g \cdot (g/h^{r_1+1})^{p_1}$ is a G-invariant germ of a pure meromorphic function.

Sketch of proof

- Arguing inductively along the Euclid's sequences of (p, q) and $(\overline{p}, \overline{q})$ one can always construct a G-invariant pure meromorphic function unless $r_j = \overline{r}_j$ for all $j = 1, \cdots, n+1$. But this means that $(p, q) = (\alpha s_n, \beta s_n)$ and $(\overline{p}, \overline{q}) = (\alpha \overline{s}_n, \beta \overline{s}_n)$ for some $\alpha, \beta \in \mathbb{Z}_+$. Therefore g, h, and f are powers of the same holomorphic function $f^{s_n}(\overline{f})^{\overline{s}_n}$, thus f and g cannot be generically transverse. A contradiction! The reasoning in the case of many irreducible factors is analogous, being in fact a consequence of the above reasoning.

A straightforward consequence is the following:

Corollary

Let $X \in \mathrm{Gen}(\mathfrak{X}(\mathbb{C}^3, 0))$ and S_X be the distinguished axis of X. Suppose that $\mathcal{F}(X)$ admits a meromorphic first integral, then the holonomy group $\mathrm{Hol}(\mathcal{F}(X), S_X, \Sigma)$ is periodic.

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Non periodic groups with finite orbits

Example

Blowing up $G = (g_1, g_2) = (x + y^2, y)$ at the origin one has

$$\widetilde{G}(t, x) = (t - t^3 x + t^5 x^2 - t^7 x^3 + \cdots, x + tx)$$

whose orbits are finite and confined in the level sets of $\widetilde{f}(t, x) = tx$ (In fact, G acts in these level sets of $\widetilde{f}$ in some sort of translation whose orbits increase in cardinality as $\widetilde{f}(t, x) \to 0$). From Proposition 11 G does not preserve the level sets of a couple of generically transverse functions $f, g \in \mathcal{O}_2$.

Non periodic holonomy with finite orbits

Example

Let $X(x) = -[x_1 - x_2^2(x_3)^2/2\pi i]\frac{\partial}{\partial x_1} - 3x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}$, then $S := \{x_1 = x_2 = 0\}$ is invariant by X and the holonomy of $\mathcal{F}(X)$ with respect to S evaluated at $\Sigma = (x_3 = 1)$ has the form

$$h(x_1, x_2) = (x_1 + x_2^2, x_2).$$

Non periodic holonomy with finite orbits

Completing the above example we obtain:

Example

Consider the vector field $X(x,y,z) = -[x - \frac{1}{2\pi \mathbf{i}} y^2 z^2]\frac{\partial}{\partial x} - 3y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$, after one blow up along the z-axis one has

$$\pi^* X(t,x,z) = -x(1 - \frac{1}{2\pi \mathbf{i}} t^2 x z^2)\frac{\partial}{\partial x} - t(2 - t^2 x z^2)\frac{\partial}{\partial x_2} + z\frac{\partial}{\partial z}$$

which has an isolated singularity at the origin, and whose holonomy with respect to the z-axis is precisely the map $\widetilde{G}$ in Example 13. Thus it satisfies condition $(\star)$ and has all leaves closed but does not admit a first integral in the sense of Definition 1.

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We consider a germ $X \in \text{Gen}(\mathfrak{X}(\mathbb{C}^3, 0))$ satisfying condition (*).

Definition (stability)

The germ X is *transversely stable* with respect to S_X if for any representative X_U of the germ X, defined in an open neighborhood U of the origin, any open section $\Sigma \subset U$ transverse to S_X with $\Sigma \cap S_X = \{q_\Sigma\} \neq \{0\}$, and any open set $q_\Sigma \in V \subset \Sigma$ there is an open subset $q_\Sigma \in W \subset V$ such that all orbits of X_U through W intersect Σ only in V.

Lemma

Let $G \in \mathrm{Diff}(\mathbb{C}^2, 0)$ *be represented by the map* $G\colon W \to V$*, where* $W \subset V$ *are open neighborhoods of the origin with compact closure. Suppose* G *has finite orbits with stable positive semiorbit, i.e., there are* W *and* V *as above with* $W \subset V$ *and satisfying* $G^{\circ(n)}(x) \subset V$ *for all* $x \in W$ *and* $n \in \mathbb{Z}_+$*. Then* G *is periodic, i.e., there is* $p \in \mathbb{Z}_+$ *such that* $G^{\circ p} = \mathrm{Id}$*.*

Theorem

Suppose that $X \in \mathrm{Gen}(\mathfrak{X}(\mathbb{C}^3, 0))$. *Then the following conditions are equivalent:*

1. $\mathcal{F}(X)$ *has a holomorphic first integral.*
2. X *satisfies condition* $(\star)$, *the leaves of* $\mathcal{F}(X)$ *are closed off the origin and transversely stable with respect to* S_X.

Corollary

Let $X, Y \in \mathrm{Gen}(\mathfrak{X}(\mathbb{C}^3, 0))$ *be generic germs of holomorphic vector fields, both satisfying condition* $(\star)$. *Assume that* X *and* Y *are topologically equivalent. Then* X *has a holomorphic first integral if and only if* Y *admits a holomorphic first integral.*

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Theorem

Suppose that $X \in \text{Gen}(\mathfrak{X}(\mathbb{C}^3, 0))$ *satisfies condition* $(\star)$ *and let* S_X *be the distinguished axis of* X*. Then the following conditions are equivalent:*

1. *The leaves of* $\mathcal{F}(X)$ *are closed off the origin and transversely stable with respect to* S_X*;*
2. $\text{Hol}(\mathcal{F}(X), S_X, \Sigma)$ *has finite orbits and is (topologically) stable;*
3. $\text{Hol}(\mathcal{F}(X), S_X, \Sigma)$ *is periodic;*
4. $\mathcal{F}(X)$ *has a holomorphic first integral.*
5. *The leaves of* $\mathcal{F}(X)$ *are closed off the origin and there is an adapted flag* $(\mathcal{F}(X), \mathfrak{F}(\omega))$*;*
6. *The leaves of* $\mathcal{F}(X)$ *are closed off the origin and there is a flag* $\mathcal{F}(X) \subset \mathfrak{F}(\omega)$ *such that* $\mathfrak{F}(\omega)$ *is a Kupka component of radial type.*

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- Marco Brunella. *A global stability theorem for transversely holomorphic foliations*. Ann. Global Anal. Geom. 15 (1997), no. 2, 179–186.
- Marco Brunella. *Inexistence of invariant measures for generic rational differential equations in the complex domain*. Bol. Soc. Mat. Mexicana (3), 2006.
- Marco Brunella; Marcel Nicolau. *Sur les hypersurfaces solutions des équations de Pfaff.* C. R. Acad. Sci. Paris Sér. I Math. 329 (1999), no. 9, 793–795.
- F.-E. Brochero-Martinez. *Groups of germs of analytic diffeomorphisms in* $(\mathbb{C}^2, 0)$. Journal of Dynamical and Control Systems, Vol. 9, No. 1, 2003, 1-32.
- W. Burnside. *On criteria for the finiteness of the order of a group of linear substitutions*, Proc.London Math. Soc. (2) 3 (1905), 435-440.

- Claude Godbillon. *Feuilletages*: Études géométriques. Progress in Mathematics, 98. Birkhäuser Verlag, Basel, 1991. xiv + 474 pp.
- E. Ghys, *Holomorphic Anosov systems*. Invent. Math. 119, 585-614 (1995).
- E. Ghys. *À propos d'un théorème de J.-P. Jouanolou concernant les feuilles fermées des feuilletages holomorphes*. Rend. Circ. Mat. Palermo (2) 49 (2000), no. 1, 175–180.
- Jean-Pierre Jouanolou. *Équations de Pfaff algèbriques*; Lecture Notes in Math. 708, Springer-Verlag, Berlin, 1979.
- J.-F. Mattei & R. Moussu, *Holonomie et intégrales premiéres*, Ann. Sci. École Norm. Sup. (4) **13** (1980), 469–523.
- Fábio Santos; Bruno Scardua. *Stability of complex foliations transverse to fibrations*, to appear in Proceedings of the American Mathematical Society.

- Georges Reeb. *Variétés feuilletées, feuilles voisines*; C.R.A.S. Paris 224 (1947), 1613-1614.
- I. Schur. *Über Gruppen periodischer substitutionen*, Sitzungsber. Preuss. Akad. Wiss. (1911), 619–627.
- B. Scárdua; *Complex Projective Foliations Having Subexponential Growth*, Indagationes Math. N.S. 12 (3) pp. 293-302 Sept. 2001.
- B. Azevedo Scárdua, *On the existence of stable compact leaves for transversely holomorphic foliations*, pre-print 2011, submitted, arXiv:1204.0095v1 [math.GT].
- B. Azevedo Scárdua; *Integration of complex differential equations*. Journal of Dynamical and Control Systems, issue 1, vol. **5**, pp.1-50, 1999.