Folding maps on a crosscap

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Abstract
We study the singularities of the family of folding maps on a crosscap. We give a list of the generic singularities that appear in the members of the family, and characterise them geometrically.

Introduction

Given a plane W in Euclidean space $\mathbb{R}^3$ with normal vector η, the **folding map** with respect to W is the map $f:\mathbb{R}^3\to\mathbb{R}^3$ given by

$$f(p)=q+\lambda^2\eta,$$

where q is the projection of p into W along η. Thus, p and its reflection with respect to W have the same image by f (see **Figure 1**).

Let Z a 3-manifold such that Z parametrizes the planes in $\mathbb{R}^3$. We define the **family of folding maps**

$$G:\mathbb{R}^3\times Z\to\mathbb{R}^3$$

by $G(p,z)=f_z(p)$, where $f_z(p)$ is the folding map with respect to the plane determined by z.

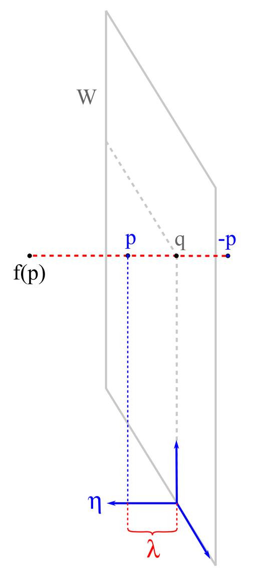

Figure 1: Folding map.

Given an embedding $g:X\to\mathbb{R}^3$, that is a smooth surface in $\mathbb{R}^3$, we obtain a **family the folding maps on** X

$$G_g:X\times Z\to\mathbb{R}^3,$$

by restriction (see **Figure 2**).

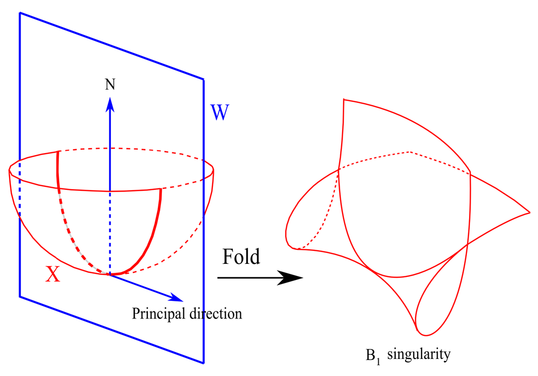

Figure 2: Folding map on a regular surface.

Bruce & Wilkinson [2, 7] studied the family of folding maps on smooth surfaces in $\mathbb{R}^3$.

In particular, they proved that the bifurcation set $\mathcal{B}(F_g)$ is the dual of the union of the focal and symmetry sets of X, where X is a smooth surface in $\mathbb{R}^3$.

Respect that duality and its corresponding geometry, we are interested in this cases:

- the subparabolic points in X correspond to S_2 singularity type of the folding map,
- the ridge points in X correspond to B_2 singularity type of the folding map.

The family of folding maps on a generic crosscap

We consider a geometric crosscap in $\mathbb{R}^3$ parametrized by $\phi:U\subset\mathbb{R}^2\to\mathbb{R}^3$, with

$$\phi(x,y)=(x,xy+p_3y^3+O(4),ax^2+bxy+y^2+\sum_{i=0}^{3}q_{3i}x^{3-i}y^i+O(4)), \quad (1)$$

see [1, 6].

In [6] is given a geometric characterization for the crosscap (1) in terms of the parameter a, namely, the crosscap is elliptic (respectively hyperbolic) if $a>0$ (respectively $a<0$) and parabolic if $a=0$ (see **Figure 3**).

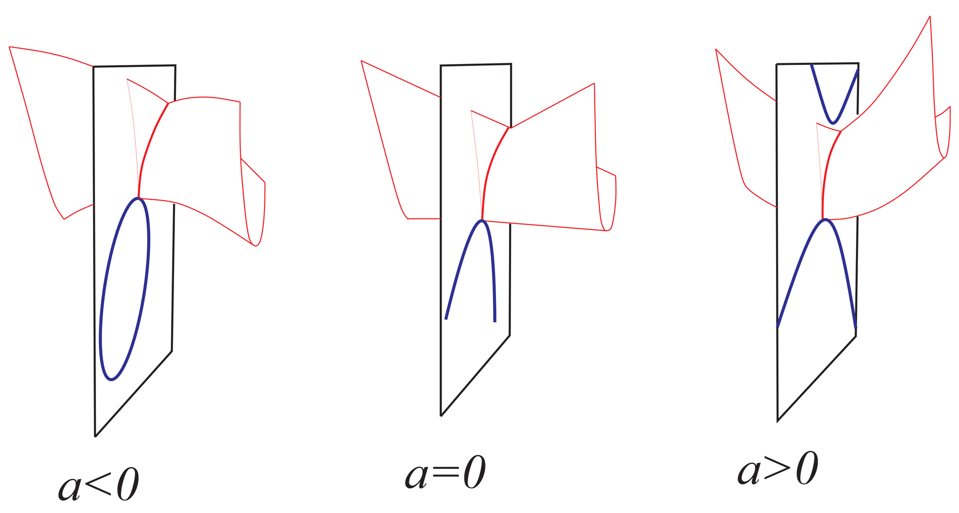

Figure 3: Classification of geometric crosscaps and their focal conics.

In [6] is also proved that the **tangent cone** T.C. to the crosscap at the crosscap point is the plane generated by the tangent direction and the limiting tangent of the double point curve.

For $\eta\in\mathbb{S}^2$ and $\delta\in\mathbb{R}$, consider the plane

$$P_{(\eta,\delta)}=\left\{p\in\mathbb{R}^3\,|\,\langle p,\eta\rangle=\delta\right\}.$$

Thus, the set of all the planes in $\mathbb{R}^3$ can be parametrized locally by $\mathbb{S}^2\times\mathbb{R}$.

We fix a generic crosscap ϕ then, the **family of folding maps on a crosscap** $G:U\times\mathbb{S}^2\times\mathbb{R}\to\mathbb{R}^3$ is given by

$$\begin{aligned}G(x,y,\eta,\delta)&=\phi(x,y)+(\langle\eta,\phi(x,y)\rangle-\delta)(\langle\eta,\phi(x,y)\rangle-\delta-1)\eta,\\&=f_{(\eta,\delta)}(x,y),\end{aligned}$$

where $f_{(\eta,\delta)}$ correspond to the folding map with respect to the plane $P_{(\eta,\delta)}$.

Proposition 1
For each $(\eta,\delta)\in\mathbb{S}^2\times\mathbb{R}$, the folding map on a crosscap $f_{(\eta,\delta)}$ is singular at the origin and the singularity is more degenerate than a crosscap if, and only if, $\delta=0$.

Denote $f_{(\eta,0)}$ by f_η and let $\eta=(\alpha,\beta,\gamma)$ with $\alpha^2+\beta^2+\gamma^2=1$.

Theorem 2
For a generic crosscap ϕ, the folding map f_η in the family G has singularities $\mathcal{A}$-equivalent to one of the following types:

***Case (i)**: η is transversal to the tangent cone*

Type	Normal form	$\mathcal{A}_e$-cod.	Condition
$B_2^{\pm}$	$(x,y^2,x^2y\pm y^5)$	2	$\alpha\neq0,\gamma\neq p_3\alpha$;
$B_3^{\pm}$	$(x,y^2,x^2y\pm y^7)$	3	$\alpha\neq0,\gamma=p_3\alpha,(*)$;
$B_4^{\pm}$	$(x,y^2,x^2y\pm y^9)$	4	$\alpha\neq0,\gamma=p_3\alpha,(**)$;
$C_3^{\pm}$	$(x,y^2,xy^3\pm x^3y)$	3	$\alpha=0,\Phi(\beta,\gamma)\neq0$;
$C_4^{\pm}$	$(x,y^2,xy^3\pm x^4y)$	4	$\alpha=0,\Phi(\beta,\gamma)=0$;
$F_{1,0}$	$(x,y^2,x^3y+A_1xy^5+B_1y^7)$	4	$\beta=1,4A_1^3+27B_1^2\neq0$;

where $\Phi(\beta,\gamma)=-2b\beta^3+(4a-b^2+2)\beta^2\gamma+\gamma^3$ *and $F_{1,0}$ is an unimodal singularity.*

Case (ii): *η is in the tangent cone and $\gamma\neq0$*

Type	Normal form	$\mathcal{A}_e$-cod.	Condition
P_3	$(x,xy+y^3,xy^2+ky^4)$	3	$\alpha\neq-p_3\gamma,k\neq\frac{1}{2},1,\frac{3}{2}$;
$P_4(\frac{1}{2})$	$(x,xy+y^3,xy^2+\frac{1}{2}y^4)$	4	$\alpha\neq-p_3\gamma$ e $\Psi(\frac{1}{2},\alpha)=0$;
$P_4(\frac{3}{2})$	$(x,xy+y^3,xy^2+\frac{3}{2}y^4)$	4	$\alpha\neq-p_3\gamma$ e $\Psi(\frac{3}{2},\alpha)=0$;
$P_4(1)$	$(x,xy+y^3,xy^2+y^4)$	4	$\alpha\neq-p_3\gamma$ e $\Psi(1,\alpha)=0$;
R_4	$(x,xy+y^6+A_2y^7,xy^2+y^4+B_2y^6)$	4	$\alpha=-p_3\gamma$;
T_4	$(x,xy+y^3,y^4)$	4	$\gamma=1$;

where $\Psi(k,\alpha)=4k^2(p_3^2+1)\alpha^2-4k(p_3^2+1)\alpha+1.$

Case (iii) $\alpha=1$, *and then f_η has corank 2 and is equivalent to*

$$(x^2,xy+y^3,y^2+A_3x^3+B_3x^2y+C_3xy^2+y^3).$$

The **Figure 4** shows the stratification of the parameter space associate with the **Theorem 2**.

Proposition 2
The singularities of the family of folding maps on a crosscap are not versally unfolded by the family G.

The geometry of the folding maps

There is a geometric correspondence for some of the singularities of the folding family. For this part we use the definitions for subparabolic and ridge points given in [3].

We say that $p\in\phi$ is a subparabolic point relative to v_i if $v_ik_j(p)=0$, $i\neq j$, where v_i is a principal direction and k_i is a principal curvature. Analogously, $p\in\phi$ is a ridge point relative to v_i if $v_ik_i(p)=0$.

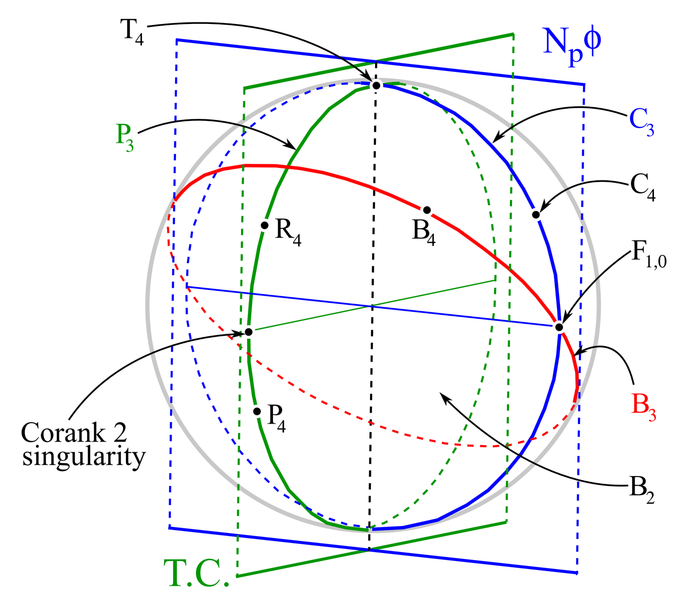

Figure 4: Stratification of the parameters space.

Theorem 3
We have the following characterization of the singularities of the folding maps on a crosscap.
(i) *The subparabolic points relative to v_2 correspond to C_4 singularities.*
(ii) *The ridge points relative to v_1 correspond to $F_{1,0}$ and T_4 singularities.*
(iii) *The corank 2 map germ appears when η is parallel to the tangent direction.*
(iv) *When η is in the limiting tangent of the double point curve the singularity is of type R_4.*

For ϕ consider the family of distance squared functions $D:\phi\times\mathbb{R}^3\to\mathbb{R}$ given by

$$D(p,h)=D_h(p)=\langle p-h,p-h\rangle\,.$$

It is well known (see for example [4]) that the focal set can be modelled locally by the bifurcation set of D.

Is showed in [6] that the part of the focal set corresponding to the crosscap point is a conic section in the normal space $N_{(0,0,0)}\phi$ (see **Figure 3**).

Proposition 3
The tangent space to the focal set of ϕ at points on the focal conic is constant and it coincide with the normal space at the crosscap point

References

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Acknowledgements

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