Affine Focal Sets of Codimension 2 Submanifolds contained in Hypersurfaces

Marcos Craizer

Catholic University of Rio de Janeiro

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Outline

Affine Differential Geometry

Curves contained in Surfaces in the Affine 3-Space

Envelope of Tangent Spaces of a Submanifold $N^n \subset M^{n+1} \subset \mathbb{R}^{n+2}$

Affine Geometry of a Submanifold $N^n \subset M^{n+1} \subset \mathbb{R}^{n+2}$

Parallel Darboux Vector Fields

Affine Focal Set of an Immersion $N \subset M$

Some Classes of Immersions

Umbilic Immersions

Next Section

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Affine Differential Geometry

The Affine Differential Geometry of hypersurfaces is classical: Tzitéica (1908), Blaschke, Radon, Pick, Berwald, Thomsen (1916-1923), Cartan (1924), Kubota, Süss, Su Buchin, Nakajima ($\approx$ 1930).

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For higher codimensions, there are much less theory: Burstin-Mayer (1927), Weise (1939), Klingenberg (1951), Nomizu-Vrancken (1993).

References:

- Buchin, S.: Affine Differential Geometry, 1983.
- Nomizu, K., Sasaki, T.: Affine Differential Geometry, 1994.

Codimension 2 submanifolds contained in a hypersurface

We shall discuss here the case of codimension 2 submanifolds N contained in hypersurface M.

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In fact, we need only the hypersurface M around the submanifold N, or equivalently, the submanifold N together with a tangent space (to M) at each point.

We can also think of submanifolds N together with a distinguished transversal vector field ξ (tangent to M).

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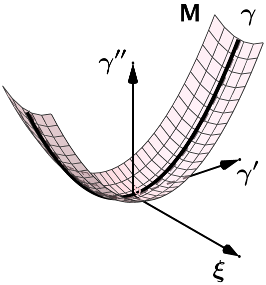
Assume that the osculating plane of γ does not coincide with the tangent plane of M. There exists a unique direction ξ tangent to M and transversal to γ such that $D_X\xi$ is tangent to M, for any X tangent to γ. This direction is called the (osculating) Darboux direction of $\gamma \subset M$.



Affine metric and affine normal plane

There exists a unique vector field $\xi(t)$ in the Darboux direction which is parallel, i.e., $\xi'(t)$ is tangent to $\gamma(t)$, $t \in I$.

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$$[\gamma'(t), \gamma''(t), \xi(t)] = 1.$$

Then $\gamma'''(t)$ is tangent to M.

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The metric $g = dt$ is called affine metric and the plane bundle $\{\gamma''(t), \xi(t)\}$ is called affine normal plane bundle of $\gamma \subset M$.

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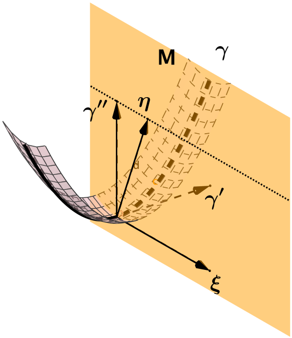
The Darboux-Frenet equations for the frame $\{\gamma'(t), \gamma''(t), \xi(t)\}$ are

$$\begin{cases} (\gamma')' = \gamma'', \\ (\gamma'')' = -\rho\gamma' + \tau\xi, \\ \xi' = -\sigma\gamma'. \end{cases}$$

A parallel basis for the affine normal plane

Choose $\lambda(t)$ such that $\lambda'(t) = -\tau(t)$. Observe that λ may not be globally defined for closed curves. Define

$$\eta(t) = \gamma''(t) + \lambda(t)\xi(t).$$



New Darboux-Frenet equations

We have

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$$\begin{cases} \gamma'' = \eta - \lambda\xi, \\ \eta' = -\mu\gamma', \\ \xi' = -\sigma\gamma'. \end{cases}$$

Envelope of Tangent Planes

The Envelope of Tangent Planes of M along γ is parameterized by

$$\phi(t, u) = \gamma(t) + u\xi(t), \quad t \in I, \; u \in \mathbb{R},$$

where $\xi(t)$ is the Darboux direction. It is also called Osculating Developable Surface.

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The Envelope of Tangent Planes reduces to a point if and only if $\sigma = -1$, constant, and $\xi = \gamma$ (centro-affine geometry). This case is of particular interest in Computer Vision, since the curves are silhouette curves or visual contours of an object. They are also called non-brightening curves (K. Saji).

Singularities of the Envelope of Tangent Planes

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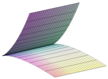
2) If $\sigma(t) \neq 0$, $u = \sigma^{-1}(t)$ and $\sigma'(t) \neq 0$, then ϕ is locally diffeomorphic to a cuspidal edge.



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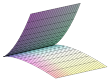
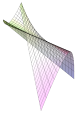


3) If $\sigma(t) \neq 0$, $u = \sigma^{-1}(t)$, $\sigma'(t) = 0$ and $\sigma''(t) \neq 0$, then ϕ is locally diffeomorphic to a swallowtail.



Affine Focal Sets

The equation of the affine normal plane at $\gamma(t)$ is $F(x,t) = 0$, where

$$F(x,t) = [x - \gamma(t), \eta(t), \xi(t)]$$

The envelope of the affine normal planes is the set

$$\mathcal{B} = \{x \in \mathbb{R}^3 |\ F = F_t = 0,\ \textit{for some}\ t \in I\}.$$

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Observe that $\mathcal{B}$ is also the bifurcation set of the affine distance function

$$\Delta(x,t)=\left[x-\gamma(t),\gamma'(t),\xi(t)\right].$$

The set $\mathcal{B}$ is also called affine focal set or evolute of the immersion $\gamma\subset M$.

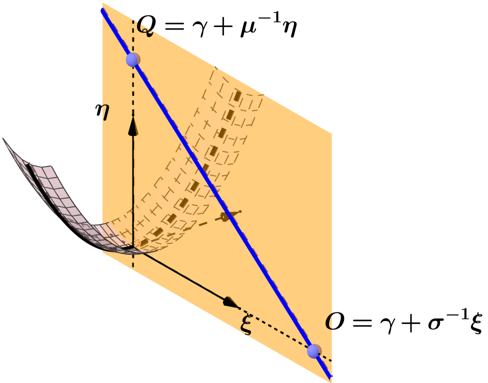
The affine focal set is a developable surface

We have that

$$\mathcal{B} = \{l(t)|\ t \in I\},$$

where $l(t)$ denote the line connecting

$$O(t) = \gamma(t) + \sigma^{-1}(t)\xi(t), \quad Q(t) = \gamma(t) + \mu^{-1}(t)\eta(t).$$



Singularities of the Affine Focal Set

The equation of the affine normal plane is

$$F(x,t) = [x - \gamma(t), \eta(t), \xi(t)] .$$

Then $F = F_t = 0$ if and only if $x = \gamma(t) + u\eta(t) + v\xi(t)$ and

$$u\mu + v\sigma = 1.$$

If $F = F_t = 0$, then

$$F_{tt} = u\mu' + v\sigma'.$$

If $F = F_t = F_{tt} = 0$ then

$$F_{ttt} = u\mu'' + v\sigma''.$$

Finally if $F = F_t = F_{tt} = F_{ttt} = 0$, then

$$F_{tttt} = u\mu''' + v\sigma'''.$$

Singularities of the Affine Focal Set

Theorem

Let $\mathcal{B}$ be the affine focal set of $\gamma \subset M$. Each point of $\mathcal{B}$ at $\gamma(t)$ belongs to the line

$$u\mu + v\sigma = 1.$$

Then

1. *$\mathcal{B}$ is smooth if $u\mu' + v\sigma' \neq 0$.*
2. *$\mathcal{B}$ is locally diffeomorphic to a cuspidal edge if $u\mu' + v\sigma' = 0$ and $u\mu'' + v\sigma'' \neq 0$.*
3. *$\mathcal{B}$ is locally diffeomorphic to a swallowtail if $u\mu'' + v\sigma'' = 0$ and $u\mu''' + v\sigma''' \neq 0$.*

Immersions whose affine focal set is a single line

The affine focal set $\mathcal{B}$ reduces to a single line if and only if σ and μ are constants.

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The affine focal set $\mathcal{B}$ reduces to a single line if and only if σ and μ are constants. Assuming $\sigma = -1$, we obtain $\xi = \gamma$ and so

$$\gamma'''(t) = -\rho(t)\gamma'(t) + \tau(t)\gamma(t).$$

The condition $\mu' = 0$ can be written as $\tau = -\rho'$. Thus

$$\gamma'''(t) = -\left(\rho(t)\gamma(t)\right)',$$

or equivalently,

$$\gamma''(t) = -\rho(t)\gamma(t) + Q,$$

for some constant vector Q.

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for some constant vector Q.

Assume $Q = (0,0,1)$ and write $\gamma = (\psi, z)$. Then

$$\psi''(t) = -\rho(t)\psi(t); \quad z''(t) = -\rho(t)z(t) + 1.$$

Immersions whose affine focal set $\mathcal{B}$ is a single line

Theorem: (C., M.J.Saia, L.Sánchez) For a planar curve Ψ, denote $\psi = \Psi'$ and $z(t) = [\Psi(t) - O, \psi(t)]$ (the affine distance or support function of Ψ with respect to an origin O). Then the affine focal set of the spacial curve

$$\gamma(t) = (\psi(t), z(t))$$

is a single line. Conversely, any curve $\gamma \subset M$ whose affine focal set is a single line is obtained by this construction.

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Since $\Psi''' = -\rho\Psi'$, ρ is the affine curvature of the planar curve Ψ. Moreover, $\rho' + \tau = 0$ and so $\rho' = 0$ if and only if $\gamma(t)$ is flat.

Corollary: Closed curves contained in surfaces whose affine focal set is a single line have at least six flat points.

A projectively invariant six vertex theorem

In the centro-affine case ($\sigma = -1$, $\xi = \gamma$), write

$$\gamma'''(t) = -\rho(t)\gamma'(t) + \tau(t)\gamma(t),$$

$$h(t) = \rho'(t) + 2\tau(t).$$

The cubic form $h(t)dt^3$ is projectively invariant and $h(t)^{1/3}dt$ is called the projective arc-length.

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If we represent γ by a planar curve, $\tau = 0$ and ρ is the affine curvature of γ. We conclude that any closed curve admit at least six points where $h(t) = 0$. Geometrically, this means that γ has higher order contact with a quadratic cone at least six times.

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Darboux direction

For $N^n \subset M^{n+1} \subset \mathbb{R}^{n+2}$, take $\{X_1, ..., X_n\}$ a local frame of N, ξ tangent to M and transversal to N and η transversal to M. Write

$$D_X Y = \nabla_X Y + h^1(X, Y)\xi + h^2(X, Y)\eta.$$

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Under the non-degeneracy condition, there exists a unique direction ξ tangent to M such that $D_X\xi$ is tangent to M, for any $X \in TN$.

We shall call this direction the *Darboux direction* of $N \subset M$.

Envelope of Tangent Spaces

Let $\{X_1, .., X_n\}$ be a frame for TN. The tangent space of M at $p \in N$ is given by $F = 0$, where

$$F(x) = [x - p, X_1, ..., X_n, \xi] .$$

The Envelope of Tangent Spaces of $N \subset M$ is given by

$$ET_N(p, u) = p + u\xi(p), \quad p \in N, \; u \in \mathbb{R}.$$

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The Envelope of Tangent Spaces of $N \subset M$ is given by

$$ET_N(p, u) = p + u\xi(p), \quad p \in N, \ u \in \mathbb{R}.$$

Write

$$D_X\xi = -S_\xi X + \tau_1^1(X)\xi$$

Then $ET_N(p, u)$ is smooth if $u \neq \sigma^{-1}$, for some non-zero eigenvalue σ of the shape operator S_ξ.

Envelope of Tangent Spaces- Simple Singularities

We show through examples that any simple singularity can appear in ET_N. We recall that any simple singularity is $\mathcal{R}$-equivalent to A_k, $k \geq 2$, D_k, $k \geq 4$, E_6, E_7 or E_8. (*)

(*) Equiaffine Darboux frames for codimension 2 submanifolds contained in hypersurfaces, M.Craizer, M.J.Saia, L.Sánchez, J.Math.Soc.Japan, 2016.

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Consider $M \subset \mathbb{R}^{n+2}$ given by the graph of $f(t, y)$, $t = (t_1, ..., t_n)$. We shall assume that $f = f_{t_i} = f_y = 0$ at the origin, for any $1 \leq i \leq n$. Let N be the submanifold $y = g(t)$ and assume that $g_{t_i} = 0$ at $t = 0$, i.e., the tangent space of N is generated by $\{e_i\}$, $1 \leq i \leq n$.

(1) Let

$$f(t,y) = \frac{t^2}{2} + \frac{1}{6}t^3 + \frac{\sigma}{2}t^2 y, \quad g(t) = 0,$$

Then, close to $(0, \sigma^{-1}, 0)$,

$$F(t, x_1, x_2 + \sigma^{-1}, x_3) = -\frac{1}{3}t^3 + \frac{\sigma}{2}t^2 x_2 + (\frac{1}{2}t^2 + t)x_1 - x_3,$$

which is a versal unfolding of an A_2 point.

(2) Let

$$f(t,y) = \frac{t^2}{2} + \frac{1}{24}t^4 + \frac{\sigma}{2}t^2 y, \quad g(t) = 0,$$

Close to $(0, \sigma^{-1}, 0)$,

$$F(t, x_1, x_2 + \sigma^{-1}, x_3) = -\frac{1}{8}t^4 + \frac{\sigma}{2}t^2 x_2 + (\frac{1}{6}t^3 + t)x_1 - x_3,$$

which is a versal unfolding of an A_3 point.

(3) For general $k \geq 3$, let $\sigma = 1$, $t = (t_1, ..., t_{k-2})$, i.e., $n = k - 2$,

$$f(t, y) = \frac{1}{2}|t|^2 + \frac{1}{2}t_1^2 y + \sum_{j=2}^{k-2} t_1^{j+1} t_j,$$

$$g(t) = -t_1^{k-1} - \sum_{j=2}^{k-2}(j+1)t_1^{j-1} t_j$$

Then, close to $(0, ..., 1, 0)$,

$$F = t_1^{k+1} - \frac{1}{2}\sum_{j=2}^{k-2} t_j^2 + x_1(t_1 - t_1^k) + \sum_{j=2}^{k-2} x_j(t_j + t_1^{j+1}) + \frac{1}{2}t_1^2 x_{k-1} - x_k.$$

which is a versal unfolding of an A_k point.

(4) For a general $k \geq 4$, take

$$f = \frac{1}{2}|t|^2 + \frac{y}{2}(t_1^2 + t_2^2) + t_1^{k-1} + t_1 t_2^2 + \sum_{j=3}^{k-2} t_1^j t_j + \sum_{j=3}^{k-2} t_1^{j-2} t_2^2 t_j$$

and $g = -\sum_{j=3}^{k-2} j t_j t_1^{j-2}$. Long but straightforward calculations show that, close to $(0, ..., 1, 0)$,

$$F = (2-k)t_1^{k-1} - 2t_1 t_2^2 - \frac{1}{2}\sum_{j=3}^{k-2} t_j^2 - x_k + \frac{1}{2}(t_1^2 + t_2^2)x_{k-1}$$

$$+\sum_{j=3}^{k-2} x_j(t_1^j + t_1^{j-2}t_2^2 + t_j) + x_2\left(t_2 + 2t_1 t_2 + \sum_{j=3}^{k-2}(2-j)t_1^{j-2}t_2 t_j\right)$$

$$+x_1\left(t_1 + (k-1)t_1^{k-2} + t_2^2 + \sum_{j=3}^{k-2}(j-2)t_1^{j-3}t_2^2 t_j\right),$$

which is a versal unfolding of a D_k point.

(5) Consider

$$f = \frac{1}{2}|t|^2 + \frac{1}{2}(t_1^2 + t_2^2)y + t_1^3 + t_2^4 + t_1 t_2 t_3 + 2t_1 t_2 t_3 y + t_1 t_2^2 t_4 + 3t_1 t_2^2 t_4 y$$

and $g = 0$. Then

$$F = -2t_1^3 - 3t_2^4 - \frac{1}{2}(t_3^2 + t_4^2) - x_6 + x_4(t_1 t_2^2 + t_4) + x_3(t_1 t_2 + t_3)$$

$$+x_1 \left(t_1 + 3t_1^2 + t_2^2 t_4 + t_2 t_3\right) + x_2 \left(t_2 + 4t_2^3 + t_1 t_3 + 2t_1 t_2 t_4\right)$$

$$+x_5 \left(\frac{1}{2}(t_1^2 + t_2^2) + 2t_1 t_2 t_3 + 3t_1 t_2^2 t_4\right)$$

which is a versal unfolding of an E_6 point.

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Umbilic Immersions

Affine metric associated with a vector field

Let ξ be a fixed vector field in the osculating Darboux direction. For a local frame $\{X_1, ..., X_n\}$ of TN and $X, Y \in TN$ define

$$G(X, Y) = [X_1, .., X_n, D_X Y, \xi].$$

Then

$$g(X, Y) = \frac{G(X, Y)}{\det G(X, Y)^{\frac{1}{n+2}}}$$

is a non-degenerate metric in N, called affine metric (*).

(*) Luis F. Sánchez: *Surfaces in 4-space from the affine differential viewpoint*, Ph.D. thesis, 2014. Advisors: M.J.Saia and J.J.Nuño-Ballesteros.

Affine normal plane bundle

There exists a vector field η transversal to M such that

1. For any $X \in TN$, $D_X\eta$ is tangent to M.
2. For any g-orthonormal frame $\{X_1, ..., X_n\}$ of TN.

$$[X_1, ..., X_n, \eta, \xi] = 1.$$

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The transversal vector field η satisfying the above conditions is not unique, any vector field of the form

$$\bar{\eta} = \eta + \lambda\xi$$

satisfies the same conditions. But, up to these transformations, it is unique. The transversal bundle $\mathcal{S}\{\xi, \eta\}$ is called the affine normal plane bundle.

Semi-umbilic immersions

For ν in the affine normal plane bundle and $X \in TN$ write

$$D_X \nu = -S_\nu X + \nabla^\perp_X \nu,$$

where $S_\nu X$ is tangent to N and $\nabla^\perp_X \nu$ belongs to the affine normal plane. The linear map S_ν is called shape operator and $\nabla^\perp_X \nu$ is called affine normal connection. The immersion $N \subset M$ is semi-umbilic (umbilic) if $S_\nu = \lambda Id$, for some (any) vector field ν in the affine normal plane bundle.

Semi-umbilic immersions

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Proposition: (J.J.Nuño-Ballesteros, L.Sánchez) If $N \subset M$ is semi-umbilic at $p \in N$, the shape operators S_ν at p commute. The converse holds if $n = 2$.

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Transon planes

Consider a point p in a surface M and a tangent vector $T \in T_pM$. A very classical result of A. Transon (1841) says that the affine vectors at p of all sections of M containing T belongs to a plane $A(p, T)$. This plane is called the Transon plane.

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This result can be generalized to hypersurfaces M by considering sections containing a hyperplane $H \subset T_pM$.

Theorem: (*) Consider and immersion $N \subset M$ and denote by H the tangent space of N at p. The affine normal plane $A(p, \xi)$ coincides with the Transon plane $A(p, H)$ if and only if ξ is parallel.

(*) Equiaffine Darboux frames for codimension 2 submanifolds contained in hypersurfaces, M.Craizer, M.J.Saia, L.Sánchez, J.Math.Soc.Japan, 2016.

Cubic forms and the apolarity condition

The cubic forms are defined as

$$\begin{aligned} C^1(X,Y,Z) &= (\nabla_X h^1)(Y,Z) + \tau_1^1(X)h^1(Y,Z) + \tau_2^1 h^2(Y,Z) \\ C^2(X,Y,Z) &= (\nabla_X h^2)(Y,Z) + \tau_1^2(X)h^1(Y,Z) + \tau_2^2 h^2(Y,Z) \end{aligned}$$

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$$tr_{h^2} C^2(X, \cdot, \cdot) = 0$$

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Proposition: The vector field ξ is parallel if and only if the cubic form C^2 is apolar with respect to h^2.

The Laplacian operator

Let $\phi : U \to \mathbb{R}^{n+2}$ be a parameterization of N and denote by Δ the Laplacian operator with respect to the metric g.

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Let $\phi : U \to \mathbb{R}^{n+2}$ be a parameterization of N and denote by Δ the Laplacian operator with respect to the metric g.

Proposition: $\Delta\phi$ belongs to the affine normal plane if and only if the Darboux vector field is parallel.

Sketch of proof: Write

$$D_X \phi_* Y - \phi_*(\hat{\nabla}_X Y) = \phi_*(K(X,Y)) + h^1(X,Y)\xi + h^2(X,Y)\eta,$$

where $K(X,Y) = \nabla_X Y - \hat{\nabla}_X Y$. The apolarity condition can be stated as $tr_g(K) = 0$. So

$$\Delta\phi = (tr_g h^1)\xi + n\eta.$$

Affine normal plane bundle- Discussion

When there exists a parallel Darboux vector field ξ, the affine metric and the affine normal plane bundle satisfy many properties similar to the codimension 1 case. Thus it seems to be a good choice of ξ.

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Nevertheless, a parallel Darboux vector field may not exist (C., M.J.Saia, L.Sánchez, 2015). In this case it seems also reasonable to choose other transversal plane bundles, like the Transon planes.

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Nevertheless, a parallel Darboux vector field may not exist (C., M.J.Saia, L.Sánchez, 2015). In this case it seems also reasonable to choose other transversal plane bundles, like the Transon planes.

From the point of view of Affine Focal Sets, the right choice is the affine normal plane bundle.

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Umbilic Immersions

Affine distance to M along N

Let $\phi : U \to \mathbb{R}^{n+2}$ be a parameterization of N. Define $F : U \times \mathbb{R}^{n+2} \to \mathbb{R}$ by

$$F(t,x) = [X_1(t), ..., X_n(t), \xi(t), x - \phi(t)],$$

where $X_i(t) = \phi_{t_i}(t)$.

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Proposition: The singular set of F coincides with the affine normal plane at $\phi(t)$.

Affine Focal Set of the immersion $N \subset M$

The bifurcation set of F is defined by

$$\mathcal{B} = \{x \in \mathbb{R}^{n+2} | \ \det(D_{tt}F) = 0\}.$$

where $D_{tt}F$ denotes the hessian matrix of $F(\cdot, x)$. The set $\mathcal{B}$ is also called the affine focal set of the immersion $N \subset M$.

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Proposition: If the immersion is semi-umbilic at a point, the affine focal set consists of n lines at this point.

Example: Product of two curves

Let $\alpha(u)$ and $\beta(v)$ be planar curves parameterized by affine arc-length and consider $\phi : I \times J \to \mathbb{R}^4$ given by

$$\phi(u,v) = (\alpha(u), \beta(v))\,.$$

Choose

$$\xi = \big(\alpha''(u), \beta''(v)\big)$$

as a parallel Darboux vector field and consider

$$\xi_1 = \big(\alpha''(u), 0\big)\,; \quad \xi_2 = \big(0, \beta''(v)\big)$$

as a parallel basis for the affine normal plane bundle. Then

$$\mathcal{B} = \{x = \phi + r\xi_1 + s\xi_2 |\ s = k(\alpha)^{-1} \text{ or } r = k(\beta)^{-1}\},$$

where k denotes affine curvature. At a point $\phi(u,v)$, $\mathcal{B}$ consists of two concurrent lines. Globally,

$$\mathcal{B} = E(\alpha) \times \mathbb{R}^2 \cup \mathbb{R}^2 \times E(\beta).$$

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Hyperplanar Submanifolds

If N is contained in a hyperplane H, we may choose ξ in the Darboux direction with a constant component in fixed direction transversal to H. This ξ is a parallel Darboux vector field. With this choice of ξ, g coincides with the Blaschke metric of $N \subset H$.

Proposition: The affine Blaschke normal η of $N \subset H$ belongs to the affine normal plane.

Corollary 1: η is umbilic if and only if $N \subset H$ is an affine sphere.

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Corollary 2: $N \subset M$ is umbilic if and only if $N \subset H$ is an affine sphere and the envelope of tangent spaces of $N \subset M$ is a cone.

Affine Focal Set- Simple Singularities

It is shown in (*) that all simple singularities appear for the affine focal set of hypersurfaces. Thus they also appear for the affine focal set of $N \subset M \subset \mathbb{R}^{n+2}$.

(*) D.Davis- Thesis- University of Liverpool, 2008

Visual contour submanifolds

Suppose all tangent planes along N meet at a point O. Taking

$$\xi(p) = \phi(p) - O,$$

we obtain $D_X \xi = X$. So ξ is parallel and $S_1 = -Id$.

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Proposition: $N \subset M$ is a visual contour if and only if there exists a parallel Darboux vector field that is also umbilic.

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Proposition: $N \subset M$ is a visual contour if and only if there exists a parallel Darboux vector field that is also umbilic.

This class of immersions is an object of study of the centro-affine differential geometry of codimension 2 submanifolds.

Submanifolds contained in hyperquadrics

If M is a hyperquadric and $N \subset M$ is arbitrary, take ξ h-orthogonal to TN satisfying $h(\xi, \xi) = 1$, where h is the Blaschke metric of M. Then ξ is a parallel Darboux vector field. With this ξ, g coincides with the restriction to N of the Blaschke metric of M.

Proposition: (J.J.Nuño-Ballesteros-M.J.Saia-L.Sánchez) The affine Blaschke normal $\eta = \phi - Q$ belongs to the affine normal plane, is parallel and umbilic.

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Proposition: $N \subset M$ is umbilic if and only if it is contained in a hyperplane.

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Differential equation of umbilic immersions

Assume $\xi = \phi$. The immersion $\phi \subset M$ is umbilic if there exists some constant vector $Q \neq O$ that belongs to the affine normal plane $A(t)$, for any $t \in U$. This is equivalent to say that for some (and hence any) g-orthonormal frame $\{X_1, ..., X_n\}$ of $N = \phi(U)$, we have

$$[\phi, X_1, ..., X_n, Q] = 1.$$

This is equivalent to

$$\frac{1}{n}\Delta\phi = -\lambda\phi + Q,$$

for some constant vector Q.

Affine distance to a hypersurface in the $(n+1)$-space

Consider a non-degenerate immersion $f : U \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ and fix $O \in \mathbb{R}^{n+1}$. Denote by $\nu : U \to \mathbb{R}^{n+1}_*$ the co-normal map of f. Define $\phi : U \to \mathbb{R}^{n+2}$ by

$$\phi(t) = (\nu(t), \nu(t) \cdot (f(t) - O)),$$

where $\nu(t) \cdot (f(t) - O)$ is the affine distance or support function of f with respect to the origin O.

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where $\nu(t) \cdot (f(t) - O)$ is the affine distance or support function of f with respect to the origin O.

Theorem: The immersion ϕ is umbilic. Conversely, any umbilic immersion is given by the above equation, for some immersion f and origin O.

Contact with hyperquadrics

Proposition: Assume that f is compact. Then ϕ is contained in a hyperplane if and only f is a n-dimensional ellipsoid.

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Proof. For a n-dimensional ellipsoid and O its center, the affine distance is constant. Conversely, if the affine distance is constant, then the affine evolute of f is a point and f is totally umbilic. Thus f is an affine sphere, and a compact affine sphere is an ellipsoid.

Sketch of proof

Let $\{X_1, ..., X_n\}$ be a h-orthonormal frame.

$$\phi = (\nu, \nu \cdot (f - O)), \ \phi_* X = (\nu_* X, \nu_* X \cdot (f - O)),$$

$$D_X \phi_* Y = (D_X \nu_* Y, D_X \nu_* Y \cdot (f - O)) - h(X, Y) Q.$$

Writing

$$D_X \nu_* Y = \sum_{i=1}^{n} a_i \nu_* X_i + b\nu,$$

$$(D_X \nu_* Y, D_X \nu_* Y \cdot (f - O)) = \sum_{i=1}^{n} a_i \phi_* X_i + b\phi,$$

which is tangent to M. Thus $g = h$ for the frame $\{\phi, Q\}$. Moreover

$$[\phi, \phi_* X_1, ..., \phi_* X_n, Q] = [\nu, \nu_* X_1, ..., \nu_* X_n] = 1,$$

thus proving that Q belongs to the affine normal plane.

The Laplacian of ϕ

We have proved that the immersion

$$\phi(t) = (\nu(t), \nu(t) \cdot (f(t) - O))$$

is umbilical and that the affine metric g coincides with the Blaschke metric h of f. From this we obtain

$$\left(\frac{1}{n}\Delta\nu, \frac{1}{n}\Delta\nu(t) \cdot (f(t) - O)\right) = (-\rho\nu, -\rho\nu \cdot (f - O) + 1),$$

where $0 \in \mathbb{R}^{n+1}$ and ρ is the affine mean curvature of f. We conclude that

$$\frac{1}{n}\Delta\phi = -\rho\phi + Q.$$

Proof of the converse-I

To prove the converse, assume that ϕ is umbilic and write $\phi = (\psi, z)$. Define f by the conditions

$$\psi \cdot (f - O) = z; \quad \psi_* X \cdot (f - O) = X(z),$$

for some origin $O \in \mathbb{R}^{n+1}$. These equations imply that $\psi \cdot f_* X = 0$, for any X, and so $\psi = \lambda \nu$, for some $\lambda \in \mathbb{R}$.

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Take a local frame $\{X_1, .., X_n\}$ g-orthonormal such that

$$[\phi, \phi_* X_1, ..., \phi_* X_n, Q] = 1.$$

Then

$$[\psi, \psi_* X_1, ..., \psi_* X_n] = 1.$$

Proof of the converse-II

So we have

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Proof of the converse-II

So we have

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Arguing as above, one can verify that $g(X, Y) = -\psi_* Y \cdot f_* X$. Thus

$$g(X, Y) = \lambda h(X, Y).$$

From this we conclude that

$$[\nu, \nu_* X_1, ..., \nu_* X_n] = \lambda^{n/2}.$$

Comparing with the above formula we obtain $\lambda = 1$, thus proving the theorem.

Thank you!

Obrigado!