THEOREMS OF ZARISKI - VAN KAMPEN TYPE

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(JOINT WORK WITH C. EYRAL)

1. Some motivation

We are working over $\mathbb{C}$. The study of the fundamental group of complements of algebraic varieties started from an attempt of Enriques to generalize the Riemann's work on multi-valued functions, so that they could be characterized by the branching locus, the number of sheets and the fundamental group of its complement.
Representing the surfaces in $\mathbb{P}^3$ as a d-branched cover of $\mathbb{P}^2$, the corresponding result about the fundamental group in dimension 2 is the following:

Theorem 1.1. *(Zariski - van Kampen) Let $C \subset \mathbb{P}^2$ be a reduced curve, defined by homogeneous polynomial Φ of degree d, and let $\pi : \mathbb{P}^2 \setminus C \to \mathbb{P}^1$ be the projection from point $P \notin C$. Let $F \subset \mathbb{P}^1$ be the generic fiber of π, which is complement of d points in $\mathbb{P}^{\Bbbk}$, and take d meridians $g_1, \dots, g_d$ around these points in $\mathbb{P}^1$. If Δ_Φ is the finite set of n points in $\mathbb{P}^1$ where $\pi|_C : C \to \mathbb{P}^1$ ramifies, we pick up meridians $\gamma_1, \dots \gamma_n$ in $\mathbb{P}^1 \setminus \Delta_\Phi$ arround each of them. Then we have a presentation $\pi_1(\mathbb{P}^2 \setminus C) \simeq < g_1, \dots, g_d : g_1 \dots g_d = 1, g_i^{\gamma_j} = g_i; i = 1, \dots d,$ $j = 1, \dots, n-1 >$.*

2. Some generalizations

One could look for possible generalizations of Zariski - van Kampen theorem from different points of view. Aiming to generalize the theorem in such a way that it would unify both the classical Zariski - van Kampen and second Lefschets theorems, leads to the next result, for which we need the monodromy variation operators.

Suppose $X = Y \setminus Z$, with $Y \subset \mathbb{P}^n$ closed and Z proper closed subset in Y. Take a Whitney stratification $\mathcal{S}$ such that Z is a union of strata, and a hyperplane L transverse to all strata (there is a generic choice for both). Define a pencil of hyperplanes containing L, with axis $\mathcal{A}$ transverse to $\mathcal{S}$ (i.e. to all strata in $\mathcal{S}$, which is possible by generic choice again). Then there are finitely many planes in the pencil

$L_1, \dots, L_N$ not transverse to $\mathcal{S}$, and in each L_i there is only finite set Σ_i of points where L_i is not transverse to some stratum. Define $\Sigma := \bigcup \Sigma_i$, and $A := \mathcal{A} \cap X$. In this way all hyperplanes are parametrized by $\mathbb{P}^1$, and let $\lambda_i \in \mathbb{P}^1, i = 1, \dots, N$ corresponding to the exceptional planes L_i, and $L_\lambda := L$.
Pick up base point $x_0 \in A$, and let D_i be a small disc centered at λ_i, disjoint from the other discs for all i. Take a loop in $\mathbb{P}^1$, $\omega_i := \rho_i.\partial D_i.\rho_i^{-1}$ where ρ_i connects x_0 with a point on ∂D_i, not intersecting any other ρ_j.
Finally, put $X_\lambda := X \cap L_\lambda$. By a theorem of D. Chéniot, for any i there is an isotopy $H_i : X_\lambda \times I \to \bigcup_{t \in I} X_{\omega_i(t)}$, for which $H(x,0) = id_{X_\lambda}(x)$, and for any $t \in I$, $H_{i,t}$ preserves pointwise A.

Definition 2.1. *Call $h_i(x) := H_i(x,1)$ the geometric monodromy of X_λ, relative to A above ω_i. It leaves A pointwise fixed, and induces an automorphism $h_{i\#}$ on $\pi_1(X_\lambda, x_0)$. Then*
$Var_i : \pi_1(X_\lambda, x_0) \to \pi_1(X_\lambda, x_0)$,
$[\alpha] \mapsto [\alpha]^{-1}.h_{i\#}(\alpha)$,
is called the i-th variation operator, associated to ω_i. It depends only on the class $[\omega_i]$.

The main result generalizes the following theorem of C.Eyral ([2]), giving at the same time partial answer to conjecture by D.Chéniot and C.Eyral ([1]).

Theorem 2.2. *(Eyral, 2004) If for any base point $z \in X \setminus \Sigma$ the maps $\pi_j(X \setminus \Sigma, z) \to \pi_j(X, z), j = 0, 1$ and $\pi_0(A) \to \pi_0(X_\lambda)$ are bijective, and for any $i = 1, \dots N$ the natural map $\pi_0(A) \to \pi_0(X_{\lambda_i} \setminus \Sigma_i)$ is surjective, then the natural map $\pi_k(X_\lambda, z) \to pi_k(X, z)$ is bijective for $k = 0$, an surjective for $k = 1$ whose kernel is the normal subgroup generated by $\bigcup Im(Var_i)$.*

To generalize further this result one needs the notion of relative variation operator. Here α is any path in X_λ from some point in A to x_0, the domain of the operator being homotopy set, and the target is homotopy group.

Definition 2.3. *Define*
$Var_i^{rel} : \pi_1(X_\lambda, A, x_0) \to \pi_1(X_\lambda, x_0)$,
$[\alpha] \mapsto [\alpha^{-1}].h_{i\#}(\alpha)$,
and call it the relative monodromy variation operator, associated with ω_i.

In the same way one has variation and relative variation operators on the blowing up $\widetilde{X}$ of X at A.
Then we have the following:

Theorem 2.4. *(Eyral, Petrov, 2016) Suppose for any $y_0 \in X \setminus \Sigma$ that the natural maps:*
i) $\pi_k(X \setminus \Sigma, y_0) \to \pi_k(X, y_0)$ are bijective for $k = 0, 1$;
ii) $\pi_0(A) \to \pi_0(X_\lambda)$ is surjective;
iii) $\pi_0(A) \to \pi_0(X_{\lambda_i} \setminus \Sigma_i)$ are surjective for all i.
Then for any $x_0 \in A$, $\pi_1(X_\lambda, x_0)/H \simeq \pi_1(X, x_0)$, where H is the normal subgroup generated by $\bigcup_i Im(Var_i^{rel}) \bigcup \{\omega_i,\ i = 1, \ldots, N\}$.

The main tools for the proof include the blowing up $\widetilde{X} \to X$ at A, then using the Lefschetz Hyperplane Section theorem, and that $\widetilde{X} \setminus \bigcup_i \widetilde{X}_{\lambda_i}$ is a locally trivial fibration over $\mathbb{P}^1 \setminus \{\lambda_i, i = 1, \ldots N\}$.

Remarks:
1) In the previous result of C. Eyral, in place of condition ii) is required bijectivity, and in its last claim, the normal subgroup is generated by $Im(Var_i)$.
2) The class of varieties satisfying the conditions of the theorem includes all smooth varieties and all locally complete intersections of pure dimension 2.

Acknowledgements: The author would like to thank CAPES - PNPD and PPG-MCCT, Universidade Federal Fluminense, for the financial support.

References

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2. C. Eyral, "Fundamental groups of singular quasi-projective varieties," Topology **43** (2004), no. 4, 749–764