AN ALGORITHM TO CLASSIFY THE ASYMPTOTIC SET ASSOCIATED TO A POLYNOMIAL MAPPING

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Abstract. We provide an algorithm to classify the asymptotic sets of the dominant polynomial mappings $F : \mathbb{C}^3 \to \mathbb{C}^3$ of degree 2, using the definition of the so-called "*façons*" in [2]. We obtain a classification theorem for the asymptotic sets of dominant polynomial mappings $F : \mathbb{C}^3 \to \mathbb{C}^3$ of degree 2. This algorithm can be generalized for the dominant polynomial mappings $F : \mathbb{C}^n \to \mathbb{C}^n$ of degree d, with any $(n, d) \in (\mathbb{N}^*)^2$.

1. Introduction

Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial mapping. Let us denote by S_F the set of points at which F is non proper, *i.e.*,

$$S_F = \{a \in \mathbb{C}^n \text{ such that } \exists \{\xi_k\}_{k\in\mathbb{N}} \subset \mathbb{C}^n, |\xi_k| \text{ tends to infinity and } F(\xi_k) \text{ tends to } a\},$$

where $|\xi_k|$ is the Euclidean norm of ξ_k in $\mathbb{C}^n$. The set S_F is called the asymptotic set of F. The comprehension of the structure of this set is very important by its relation with the Jacobian Conjecture. In the years 90's, Jelonek studied this set in a deep way and described the principal properties. One of the important results is that, if F is dominant, *i.e.*, $\overline{F(\mathbb{C}^n)} = \mathbb{C}^n$, then S_F is an empty set or a hypersurface [1].

Notice that it is sufficient to define S_F by considering sequences $\{\xi_k\}$ tending to infinity in the following sense: each coordinate of these sequences either tends to infinity or converges. In [2], the sequences tending to infinity such that their images tend to the points in S_F are labeled in terms of "*façons*", as follows: We rank the coordinates of ξ_k into three categories : i) the coordinates tending to infinity (this cotegories is not empty), ii) the coordinates $(x_{k,i})$ such that $\lim_{k\to\infty} x_{k,i}$ is a complex number "independant on the point a in a neighborhood a in S_F". This means that there exists the points neighbors of a in S_F and the sequences $\{\xi'_k\}$ such that $\lim_{k\to\infty} F(\xi'_k) = a'$ and $\lim_{k\to\infty} x'_{k,i} = \lim_{k\to\infty} x_{k,i}$. iii) the coordinates $(x_{k,i})$ such that $\lim_{k\to\infty} x_{k,i}$ is a complex number "dependant on the point a". This means that there not exists the such points a' neighbors of a in S_F. The example 2.5 illustrates these three categories.

We define a "*façon*" of the point $a \in S_F$ as a (p, q)-tuple $(i_1, \ldots, i_p)[j_1, \ldots, j_q]$ of integers where x^a_{k,i_r} tends to infinity for $r = 1, \ldots, p$ and, for $s = 1, \ldots, q$, the sequence x^a_{k,j_s} tends to a complex number independently on the point a when a describes locally S_F (definition 2.7).

The aim of this paper is to provide an algorithm to classify the asymptotic sets of dominant polynomial mappings $F : \mathbb{C}^3 \to \mathbb{C}^3$ of degree 2, using the definition of "*façons*" in [2], and then generalize this algorithm for the general case. One important tool of the algorithm is the notion of *pertinent variables*. The idea of the notion of pertinent variables is the following: Let $F = (F_1, F_2, F_3) : \mathbb{C}^3 \to \mathbb{C}^3$ be a dominant polynomial mapping of degree 2 such that $S_F \neq \emptyset$. We fix a *façon* κ of F and assume that $\{\xi_k\}$ is a sequence tending to infinity with the *façon* κ such that $F(\xi_k)$ tends to a point of S_F. Since the degree of F is 2 then each coordinate F_1, F_2 and F_3 of F is a linear combination of $f_1 = x_1$, $f_2 = x_2$, $f_3 = x_3$, $f_4 = x_1x_2$, $f_5 = x_2x_3$ and $f_6 = x_3x_1$. We call a *pertinent variable of F with respect to the façon κ* a *minimum* linear combination of $f_1, \ldots, f_6$ such that the image of the sequence $\{\xi_k\}$ by this combination does not tend to infinity (see definition 3.1).

Moreover, if F is dominant then by Jelonek, the set S_F has pure dimension 2 (see theorem 2.4). With this observation and with the idea of pertinent variables, we:

- Make the list $(\mathcal{L})$ of all possible *façons* for a polynomial mappings $F : \mathbb{C}^3 \to \mathbb{C}^3$. This list is finite. In fact, there are 19 possible *façons* (see the list (3.4)).
- Assume that a 2-dimensional irreductible stratum S of S_F admits l fixed *façons* in the list $(\mathcal{L})$, where $1 \leqslant l \leqslant 19$.
- Determine the pertinent variables of F with respect to these l *façons*.
- Restrict the above pertinent variables using the dominancy of F and the fact that the dimension of S is 2. We get the form of F in terms of these pertinent variables.
- Determine the geometry of S in terms of the form of F.
- Let l runs in the list $(\mathcal{L})$ for $1 \leqslant l \leqslant 19$. We get all the possible 2-dimensional irreductible strata of S_F. Since the dimension of S_F is 2, then we get the list of all possible asymptotic sets S_F.

With this idea, we provide the algorithm 3.10 to classify the asymptotic sets of dominant polynomial mappings $F : \mathbb{C}^3 \to \mathbb{C}^3$ of degree 2, and we obtain the classification theorem 4.1. This algorithm can be generalized for the general case of polynomial mappings $F : \mathbb{C}^n \to \mathbb{C}^n$ of degree d, where $n \geqslant 3$ and $d \geqslant 2$ (algorithm 5.1).

2. Dominancy, assymptotic set and "façons"

2.1. Dominant polynomial mapping.

Definition 2.1. Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial mapping. Let $\overline{F(\mathbb{C}^n)}$ be the closure of $F(\mathbb{C}^n)$ in $\mathbb{C}^n$. F is called *dominant* if $\overline{F(\mathbb{C}^n)} = \mathbb{C}^n$, *i.e.*, $F(\mathbb{C}^n)$ is dense in $\mathbb{C}^n$.

We provide here a lemma on the dominancy of a polynomial mapping $F : \mathbb{C}^n \to \mathbb{C}^n$ that we will use later on.

Lemma 2.2. *Let $F = (F_1, \ldots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a dominant polynomial mapping. Then, the coordinate polynomials $F_1, \ldots, F_n$ are independent. That means, there does not exist any coordinate polynomial F_η, where $\eta \in \{1, \ldots, n\}$, such that F_η is a polynomial mapping of the variables $F_1, \ldots, F_{\eta-1}, F_{\eta+1}, \ldots, F_n$.*

Proof. Assume that $F_\eta = \varphi(F_1, \dots, F_{\eta-1}, F_{\eta+1}, \dots, F_n)$ where $\eta \in \{1, \dots, n\}$ and φ is a polynomial. Then, the dimension of $F(\mathbb{C}^n)$ is less than n. Consequently, the dimension of $\overline{F(\mathbb{C}^n)}$ is less than n. That provides the contradiction with the fact F is dominant. ∎

2.2. **Asymptotic set.**

Definition 2.3. Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial mapping. Let us denote by S_F the set of points at which F is non-proper, *i.e.*,

$$S_F = \{a \in \mathbb{C}^n \text{ such that } \exists \{\xi_k\}_{k \in \mathbb{N}} \subset \mathbb{C}^n, |\xi_k| \to \infty \text{ and } F(\xi_k) \to a\},$$

where $|\xi_k|$ is the Euclidean norm of ξ_k in $\mathbb{C}^n$. The set S_F is called the asymptotic set of F.

Recall that, it is sufficient to define S_F by considering sequences $\{\xi_k\}$ tending to infinity in the following sense: each coordinate of these sequences either tends to infinity or converges to a finite number.

Theorem 2.4. [1] *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial mapping. If F is dominant, then S_F is either an empty set or a hypersurface.*

2.3. **"Façons".** In this section, let us recall the definition of *façons* as it appears in [2]. In order to a better understanding of the definition of *façons*, let us start by giving an example.

Example 2.5. [2] Let $F = (F_1, F_2, F_3) : \mathbb{C}^3_{(x_1,x_2,x_3)} \to \mathbb{C}^3_{(\alpha_1,\alpha_2,\alpha_3)}$ be the polynomial mapping such that

$$F_1 := x_1, \qquad F_2 := x_2, \qquad F_3 := x_1x_2x_3.$$

Notice that by the notations $\mathbb{C}^3_{(x_1,x_2,x_3)}$ and $\mathbb{C}^3_{(\alpha_1,\alpha_2,\alpha_3)}$, we want to distinguish the source space and the target space. We determine now the asymptotic set S_F by using the definition 2.3. Assume that there exists a sequence $\{\xi_k = (x_{1,k}, x_{2,k}, x_{3,k})\}$ in the source space $\mathbb{C}^3_{(x_1,x_2,x_3)}$ tending to infinity such that its image $\{F(\xi_k) = (x_{1,k}, x_{2,k}, x_{1,k}x_{2,k}x_{3,k})\}$ does not tend to infinity. Then $x_{1,k}$ and $x_{2,k}$ cannot tend to infinity. Since the sequence $\{\xi_k\}$ tends to infinity, then $x_{3,k}$ must tend to infinity. Hence, we have the three following cases:

1) $x_{1,k}$ tends to 0, $x_{2,k}$ tends to a complex number $\alpha_2 \in \mathbb{C}$ and $x_{3,k}$ tends to infinity. In order to determine the biggest possible subset of S_F, we choose the sequences $x_{1,k}$ tending to 0 and $x_{3,k}$ tending to infinity in such a way that the product $x_{1,k}x_{3,k}$ tends to a complex number α_3. Let us choose, for example $\xi_k = \left(\frac{1}{k}, \alpha_2, \frac{k\alpha_3}{\alpha_2}\right)$ where $\alpha_2 \neq 0$, then $F(\xi_k)$ tends to a point $a = (0, \alpha_2, \alpha_3)$ in S_F. We get a 2-dimensional stratum S_1 of S_F, where $S_1 = (\alpha_1 = 0) \backslash 0\alpha_3 \subset \mathbb{C}^3_{(\alpha_1,\alpha_2,\alpha_3)}$. We say that a *"façon"* of S_1 is (3)[1]. The symbol "(3)" in the *façon* (3)[1] means that the *third* coordinate $x_{3,k}$ of the sequence $\{\xi_k\}$ tends to infinity. The symbol "[1]" in the *façon* (3)[1] means that the *first* coordinate $x_{1,k}$ of the sequence $\{\xi_k\}$ tends to 0 which is a fixed complex number which does not depend on the point $a = (0, \alpha_2, \alpha_3)$ when a describes S_1. Notice that the second coordinate of the sequence $\{\xi_k\}$ tends to a complex number α_2 depending on the point $a = (0, \alpha_2, \alpha_3)$ when

a varies, then the indice "2" does not appear in the *façon* (3)[1]. Moreover, all the sequences tending to infinity such that their images tend to a point of S_1 admit only the *façon* (3)[1].

The two following cases are similar to the case 1):

2) $x_{1,k}$ tends to a complex number $\alpha_1 \in \mathbb{C}$, $x_{2,k}$ tends to 0 and $x_{3,k}$ tends to infinity: then the *façon* (3)[2] determines a 2-dimensional stratum S_2 of S_F, where $S_2 = (\alpha_2 = 0)\backslash 0\alpha_3 \subset \mathbb{C}^3_{(\alpha_1,\alpha_2,\alpha_3)}$.

3) $x_{1,k}$ and $x_{2,k}$ tend to 0, and $x_{3,k}$ tends to infinity: then the *façon* (3)[1, 2] determines the 1-dimensional stratum S_3 where S_3 is the axis $0\alpha_3$ in $\mathbb{C}^3_{(\alpha_1,\alpha_2,\alpha_3)}$.

In conclusion, we get

- the asymptotic set S_F of the given polynomial mapping F as the union of two planes $(\alpha_1 = 0)$ and $(\alpha_2 = 0)$ in $\mathbb{C}^3_{(\alpha_1,\alpha_2,\alpha_3)}$,
- all the *façons* of S_F of the given polynomial mapping F: they are three façons (3)[1], (3)[2] and (3)[1, 2].

Remark 2.6. The chosen sequence $\left\{\xi_k = \left(\frac{1}{k}, \alpha_2, \frac{k\alpha_3}{\alpha_2}\right)\right\}$ in 1) of the above example is called a *generic sequence* of the 2-dimensional irreductible component $(\alpha_1 = 0)$ (a plane) of S_F, since the image of any sequence of this type (with differents $\alpha_2 \neq 0$ and α_3) falls to a generic point of the plane $(\alpha_1 = 0)$. That means the images of all the sequences $\{\xi_k\}$ when α_2 runs in $\mathbb{C}\backslash\{0\}$ and α_3 runs in $\mathbb{C}$ cover $S_1 = (\alpha_1 = 0)\backslash 0\alpha_3$ and S_1 is dense in the plane $(\alpha_1 = 0)$. We can see easily that a generic sequence of the 2-dimensional irreductible component $(\alpha_2 = 0)$ of S_F is $\left(\alpha_1, \frac{1}{k}, \frac{k\alpha_3}{\alpha_1}\right)$ where $\alpha_1 \neq 0$. More generally, any sequence $\left\{\left(\frac{1}{k^r}, \alpha_2, \frac{k^r\alpha_3}{\alpha_2}\right)\right\}$, where $r \in \mathbb{N}\backslash\{0\}$ and $\alpha_2 \neq 0$, is a generic sequence of $(\alpha_1 = 0) \subset S_F$. Any sequence $\left\{\left(\alpha_1, \frac{1}{k^r}, \frac{k^r\alpha_3}{\alpha_1}\right)\right\}$, where $r \in \mathbb{N}\backslash\{0\}$ and $\alpha_1 \neq 0$, is a generic sequence of $(\alpha_2 = 0) \subset S_F$.

In the light of this example, we recall here the definition of *façons* in [2].

Definition 2.7. [2] Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a dominant polynomial mapping such that $S_F \neq \emptyset$. For each point a of S_F, there exists a sequence $\{\xi^a_k\} \subset \mathbb{C}^n$, $\xi^a_k = (x^a_{k,1}, \dots, x^a_{k,n})$ tending to infinity such that $F(\xi^a_k)$ tends to a. Then, there exists at least one index $i \in \mathbb{N}$, $1 \leqslant i \leqslant n$ such that $x^a_{k,i}$ tends to infinity when k tends to infinity. We define "*a façon of tending to infinity of the sequence* $\{\xi^a_k\}$", as a maximum (p, q)-tuple $\kappa = (i_1, \dots, i_p)[j_1, \dots, j_q]$ of different integers in $\{1, \dots, n\}$, such that:

i) x^a_{k,i_r} tends to infinity for all $r = 1, \dots, p$,

ii) for all $s = 1, \dots, q$, the sequence x^a_{k,j_s} tends to a complex number independently on the point a when a varies locally, that means:

ii.1) either there exists in S_F a subvariety U_a containing a such that for any point a' in U_a, there exists a sequence $\{\xi^{a'}_k\} \subset \mathbb{C}^n$, $\xi^{a'}_k = (x^{a'}_{k,1}, \dots, x^{a'}_{k,n})$ tending to infinity such that

a) $F(\xi^{a'}_k)$ tends to a',

b) $x^{a'}_{k,i_r}$ tends to infinity for all $r = 1, \ldots, p$,

c) for all $s = 1, \ldots, q$, $\lim\limits_{k\to\infty} x^{a'}_{k,j_s} = \lim\limits_{k\to\infty} x^{a}_{k,j_s}$ and this limit is finite.

ii.2) or there does not exist such a subvariety, then we define

$$\kappa = (i_1, \ldots, i_p)[j_1, \ldots, j_{n-p}],$$

where x^a_{k,i_r} tends to infinity for all $r = 1, \ldots, p$ and $\{i_1, \ldots, i_p\} \cup \{j_1, \ldots, j_{n-p}\} = \{1, \ldots, n\}$. In this case, the set of points a is a subvariety of dimension 0 of S_F.

We call a *façon* of tending to infinity of the sequence $\{\xi^a_k\}$ also a *a façon of a* or *a façon of S_F*.

3. An algorithm to stratify the asymptotic sets of the dominant polynominal mappings $F : \mathbb{C}^3 \to \mathbb{C}^3$ of degree 2

In this section we provide an algorithm to stratify the asymptotic sets associated to dominant polynominal mappings $F : \mathbb{C}^3 \to \mathbb{C}^3$ of degree 2. In the last section, we show that this algorithm can be generalized in the general case for dominant polynominal mappings $F : \mathbb{C}^n \to \mathbb{C}^n$ of degree d where $n \geqslant 3$ and $d \geqslant 2$. Let us recall that by degree of a polynomial mapping $F = (F_1, \ldots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$, we mean the highest degree of the monomials $F_1, \ldots, F_n$.

Let us consider now a dominant polynomial mapping $F = (F_1, F_2, F_3) : \mathbb{C}^3_{(x_1,x_2,x_3)} \to \mathbb{C}^3_{(\alpha_1,\alpha_2,\alpha_3)}$ of degree 2 such that $S_F \neq \emptyset$. An important step of this section is to define the notion of *"pertinent" variables* of F.

3.1. **Pertinent variables.** Let us explain at first the idea of the notion of *pertinent variables*: let $\{\xi_k\} = \{(x_{1,k}, x_{2,k}, x_{3,k})\}$ be a sequence in the source space $\mathbb{C}^3_{(x_1,x_2,x_3)}$ tending to infinity such that $F(\xi_k)$ tends to a point of S_F in the target space $\mathbb{C}^3_{(\alpha_1,\alpha_2,\alpha_3)}$. Then the image of ξ_k by any coordinate polynomial F_η, where $\eta = 1, 2, 3$, cannot tend to infinity. Notice that F_η can be written as the sum of elements of the form $F^1_\eta F^2_\eta$ such that if $F^1_\eta(\xi_k)$ tends to infinity, then $F^2_\eta(\xi_k)$ must tend to 0. In other words, if one element of the above sum has a factor tending to infinity with respect to the sequence $\{\xi_k\}$, then this element must be "balanced" with another factor tending to zero with respect to the sequence $\{\xi_k\}$. For example, assume that the coordinate sequences $x_{1,k}$ and $x_{2,k}$ of the sequence $\{\xi_k\}$ tend to infinity, then F_η cannot admit neither x_1 nor x_2 *alone* as an element of the above sum, but F_η can admit $(x_1 - \nu x_2)$, $(x_1 - \nu x_2)x_1$, $(x_1 - \nu x_2)x_2$ as elements of this sum, where $\nu \in \mathbb{C}\backslash\{0\}$. So we define

Definition 3.1. Let $F = (F_1, F_2, F_3) : \mathbb{C}^3_{(x_1,x_2,x_3)} \to \mathbb{C}^3_{(\alpha_1,\alpha_2,\alpha_3)}$ be a polynomial mapping of degree 2 such that $S_F \neq \emptyset$. Let us fix a *façon* κ of S_F. Then there exists a sequence $\{\xi_k\} \subset \mathbb{C}^3_{(x_1,x_2,x_3)}$ tending to infinity with the *façon* κ such that its image tend to a point in S_F.

An element in the list

$$
(3.2)\qquad \begin{cases} X_{h_i} = x_i, \text{ where } i = 1, 2, 3, \\ X_{h_j} = x_i + \nu_{h_j} x_j, \text{ where } i \neq j \text{ and } i, j = 1, 2, 3, \\ X_{h_r} = (x_i + \nu_{h_r} x_j) x_l, \text{ where } i \neq j \text{ and } i, j, l = 1, 2, 3, \\ X_{h_s} = x_i + \nu_{h_s} x_j x_l, \text{ where } i \neq j, j \neq k, k \neq i \text{ and } i, j, k = 1, 2, 3, \end{cases}
$$

$(\nu_{h_i}, \nu_{h_j}, \nu_{h_r}, \nu_{h_s} \in \mathbb{C}\setminus\{0\})$ is called *a pertinent variable of F with respect to the façon κ* if the image of the sequence $\{\xi_k\}$ by this element does not tend to infinity.

Remark 3.3. From now on, we will denote $X_1, \ldots, X_h$ pertinent variables of F with respect to a fixed *façon* and we write

$$F = \tilde{F}(X_1, \ldots, X_h).$$

Notice that we can also determine the pertinent variables of F with respect to a set of *façons* in the case we have more than one *façon*.

3.2. **Idea of the algorithm.** The aim of the algorithm that we present in this section is to describe the list of all possible asymptotic sets S_F for the dominant polynomial mappings $F : \mathbb{C}^3 \to \mathbb{C}^3$ of degree 2. In order to do that, we observe firstly that

- The list of all the possible *façons* of S_F for a polynomial mapping $F : \mathbb{C}^3 \to \mathbb{C}^3$ is

$$
(3.4)\qquad \begin{cases} 1) \text{ Group I} : (1, 2, 3), \\ 2) \text{ Group II} : (1, 2), (2, 3) \text{ and } (3, 1), \\ 3) \text{ Group III} : (1), (2) \text{ and } (3), \\ 4) \text{ Group IV} : (1, 2)[3], (1, 3)[2] \text{ and } (2, 3)[1], \\ 5) \text{ Group V} : (1)[2], (1)[3], (2)[1], [2](3), [3](1) \text{ and } [3](2), \\ 6) \text{ Group VI} : (1)[2, 3], (2)[1, 3] \text{ and } (3)[1, 2]. \end{cases}
$$

  This list has 19 *façons*.
- Since F dominant, then by the theorem 2.4, the set S_F has pure dimension 2.

With these observations, we will:

- assume that a 2-dimensional irreductible stratum S of S_F admits l fixed *façons* in the list (3.4), where $1 \leqslant l \leqslant 19$,
- determine the pertinent variables of F with respect to these l *façons*,
- restrict the above pertinent variables using the dominancy of F and the fact $\dim S = 2$. We get the form of F in terms of these pertinent variables,
- determine the geometry of S in terms of the form of F,
- let l runs in the list (3.4) for $1 \leqslant l \leqslant 19$. We get all the possible 2-dimensional irreductible strata S of S_F. Since the dimension of S_F is 2, then we get the list of all the possible asymptotic sets S_F of F.

The following example explains the process of the algorithm, *i.e.* how we can determine the geometry of a 2-dimensional irreductible stratum S of S_F admitting some fixed *façons*.

3.3. **Example.**

Example 3.5. Let $F : \mathbb{C}^3 \to \mathbb{C}^3$ be a dominant polynomial mapping of degree 2. Assume that a 2-dimensional stratum S of S_F admits the two *façons* $\kappa = (1, 2, 3)$ and $\kappa' = (1, 2)[3]$. That means that all the sequences tending to infinity in the source space such that their images tend to the points of S admit either the *façon* $\kappa = (1, 2, 3)$ or the *façon* $\kappa' = (1, 2)[3]$. In order to describe the geometry of S, we perform the following steps:

Step 1: Determine the pertinent variable of F with respect to the *façons* $\kappa = (1, 2, 3)$ and $\kappa' = (1, 2)[3]$:

- With the *façon* $\kappa = (1, 2, 3)$, all the three coordinate sequences of the corresponding sequence tend to infinity (*cf.* Definition 2.7). Up to a suiable linear change of coordinates, the mapping F admits the pertinent variables: x_1-x_2, $(x_1-x_2)x_1$, $(x_1-x_2)x_2$, $(x_1-x_2)x_3$, x_1-x_3, $(x_1-x_3)x_1$, $(x_1-x_3)x_2$, $(x_1-x_3)x_3$, x_2-x_3, $(x_2-x_3)x_1$, $(x_2-x_3)x_2$, $(x_2-x_3)x_3$, $x_1 - x_2x_3$, $x_2 - x_1x_3$ and $x_3 - x_1x_2$ (see definition 3.1).
- With the *façon* $\kappa' = (1, 2)[3]$, the first and second coordinate sequences of the corresponding sequence tend to infinity, the third coordinate sequence of the corresponding sequence tends to a fixed complex number. As we refer to the same mapping F, then up to the *same* suiable linear change of coordinates, the mapping F admits the pertinent variables: x_3, x_1x_3, x_2x_3, $x_1 - x_2$, $(x_1 - x_2)x_1$, $(x_1 - x_2)x_2$, $(x_1 - x_2)x_3$ and $(x_1 - x_3)x_3$.

Since S contains both of the *façons* κ and κ', then this surface S admits $x_1 - x_2$, $(x_1 - x_2)x_1$, $(x_1 - x_2)x_2$, $(x_1 - x_2)x_3$ and $(x_1 - x_3)x_3$ as pertinent variables. Let us denote by

$$X_1 = x_1 - x_2, \quad X_2 = (x_1 - x_2)x_1, \quad X_3 = (x_1 - x_2)x_2, \quad X_4 = (x_1 - x_2)x_3, \quad X_5 = (x_1 - x_3)x_3.$$

We can write

$$F = \tilde{F}(X_1, X_2, X_3, X_4, X_5). \tag{3.6}$$

Step 2: Assume that $\{\xi_k = (x_{1,k}, x_{2,k}, x_{3,k})\}$ and $\{\xi'_k = (x'_{1,k}, x'_{2,k}, x'_{3,k})\}$ are two sequences tending to infinity with the *façons* κ and κ', respectively.

A) Let us consider the façon $\kappa = (1, 2, 3)$ and its corresponding generic sequence $\{\xi_k = (x_{1,k}, x_{2,k}, x_{3,k})\}$:

- Assume that $X_1(\xi_k) = (x_{1,k} - x_{2,k})$ tends to a non-zero complex number. Since $\kappa = (1, 2, 3)$ then all three coordinate sequences $x_{1,k}, x_{2,k}$ and $x_{3,k}$ tend to infinity. Hence $X_2(\xi_k) = (x_{1,k} - x_{2,k})x_{1,k}$, $X_3(\xi_k) = (x_{1,k} - x_{2,k})x_{2,k}$ and $X_4(\xi_k) = (x_{1,k} - x_{2,k})x_{3,k}$ tend to infinity. In this case, X_2, X_3 and X_4 cannot be pertinent variables of F anymore. Then F admits only two pertinent variables X_1 and X_5, or $F = \tilde{F}(X_1, X_5)$. We can see that the dimension of S in this case is 1, that provides a contradiction with the fact that the dimension of S is 2. Consequently, $(x_{1,k} - x_{2,k})$ tends to 0.

- Assume that $(x_{1,k} - x_{3,k})$ tends to a non-zero complex number. Then $X_5(\xi_k) = (x_{1,k} - x_{3,k})x_{3,k}$ tend to infinity, hence X_5 cannot be a pertinent variable of F anymore, then $F = \tilde{F}(X_1, X_2, X_3, X_4)$. We choose a *generic* sequence $\{\xi_k\}$ satisfying the conditions: $X_{1,k} = (x_{1,k} - x_{2,k})$ tends to zero and $(x_{1,k} - x_{3,k})$ tends to a non-zero complex number, for example, $\xi_k = (k + \alpha/k, k + \beta/k, k + \gamma)$. Then $X_2(\xi_k) = (x_{1,k} - x_{2,k})x_{1,k}$, $X_3(\xi_k) = (x_{1,k} - x_{2,k})x_{2,k}$ and $X_4(\xi_k) = (x_{1,k} - x_{2,k})x_{3,k}$ tend to the same complex number $\lambda - \mu$. Combining with the fact $X_{1,k} = (x_{1,k} - x_{2,k})$ tends to zero, we conclude that the dimension of S in this case is 1, that provides a contradiction with the fact that the dimension of S is 2. Consequently, $(x_{1,k} - x_{3,k})$ tends to 0.

Then, with the *façon* κ, we have $(x_{1,k} - x_{2,k})$ and $(x_{1,k} - x_{3,k})$ tend to 0. Hence $(x_{2,k} - x_{3,k})$ also tends to 0. Let us choose a *generic* sequence $\{\xi_k\}$ satisfying these conditions, for example, the sequence $\{\xi_k = (k + \alpha/k, k + \beta/k, k + \gamma/k)\}$. We see that $X_{2,k} = (x_{1,k} - x_{2,k})x_{1,k}$, $X_{3,k} = (x_{1,k} - x_{2,k})x_{2,k}$ and $X_{4,k} = (x_{1,k} - x_{2,k})x_{3,k}$ tend to a same complex number $\lambda = \alpha - \beta$. Moreover, $X_{5,k} = (x_{1,k} - x_{3,k})x_{3,k}$ tends to $\mu = \alpha - \gamma$. So we have

$$\lim_{k\to\infty} F(\xi_k) = \tilde{F}(0, \lambda, \lambda, \lambda, \mu) \tag{3.7}$$

B) Let us consider now the façon $\kappa' = (1, 2)[3]$ and its corresponding generic sequence $\{\xi'_k = (x'_{1,k}, x'_{2,k}, x'_{3,k})\}$, we have two cases:

- If $X_1(\xi'_k) = (x'_{1,k} - x'_{2,k})$ tends to 0: So $X_4(\xi'_k) = (x'_{1,k} - x'_{2,k})x'_{3,k}$ tends to 0. We have $X_2(\xi'_k) = (x'_{1,k} - x'_{2,k})x'_{1,k}$ and $X_2(\xi'_k) = (x'_{1,k} - x'_{2,k})x'_{2,k}$ tend to a same complex number λ' and $X_5(\xi'_k) = (x'_{1,k} - x'_{3,k})x'_{3,k}$ tends to an arbitrary complex number μ'. Then in this case, we have

$$F(\xi'_k) = \tilde{F}(0, \lambda', \lambda', 0, \mu'). \tag{3.8}$$

- If $X_1(\xi'_k) = (x'_{1,k} - x'_{2,k})$ tends to a non-zero complex number $\lambda' \in \mathbb{C}$: So $X_2(\xi'_k)$ and $X_3(\xi_k)$ tend to infinity, thus X_2 and X_3 cannot be pertinent variables of F anymore. Moreover, $X_4(\xi'_k)$ tends to 0 and $X_5(\xi'_k)$ tends to an arbitrary complex number μ'. Then in this case, we have

$$\begin{array}{rcllll} F & = & \tilde{F}(X_1, & X_4, & X_5) \\ F(\xi'_k) & = & \tilde{F}(\lambda', & 0, & \mu'). \end{array} \tag{3.9}$$

In conclusion, we have two cases:

1) From (3.6), (3.7) and (3.8), we have

$$\begin{array}{rclccccc} F & = & \tilde{F}(X_1, & X_2, & X_3, & X_4, & X_5) \\ \lim_{k\to\infty} F(\xi_k) & = & \tilde{F}(0, & \lambda, & \lambda, & \lambda, & \mu) \\ \lim_{k\to\infty} F(\xi'_k) & = & \tilde{F}(0, & \lambda', & \lambda', & 0, & \mu'). \end{array} \qquad (*)$$

2) From (3.6), (3.7) and (3.9), we have

$$\begin{array}{rclcc} F & = & \tilde{F}(X_1, & X_4, & X_5) \\ \lim_{k\to\infty} F(\xi_k) & = & \tilde{F}(0, & \lambda, & \mu) \\ \lim_{k\to\infty} F(\xi'_k) & = & \tilde{F}(\lambda', & 0, & \mu'). \end{array} \qquad (**)$$

Step 3: We restrict the pertinent variables in the step 2 using the three following facts:

- κ and κ' are two *façons* of the same stratum S,
- $\dim S = 2$,
- F is dominant.

Let us consider the two cases (*) and (**) determined in the step 2:

1) F is of the form (*):
   - At first, we use the fact that κ and κ' are two *façons* of the same stratum S, then if X_i is a pertinent variable of F then both $X_i(\xi_k)$ and $X_i(\xi'_k)$ must tend to either an arbitrary complex number or zero.
   - Since the dimension of S is 2 then F must have at least two pertinent variables X_i and X_j such that the images of the sequences ξ_k and ξ'_k by X_i and X_j, respectively, tend independently to two complex numbers. In this case:
     + F must admit either $X_2 = (x_1 - x_2)x_1$ or $X_3 = (x_1 - x_2)x_2$ as a pertinent variable,
     + F must admit $X_5 = (x_1 - x_3)x_3$ as a pertinent variable.
   - Since F is dominant then F must admit at least 3 independent pertinent variables (see lemma 2.2). Then in this case, F must also admit $X_1 = x_1 - x_2$ as a pertinent variable. We see that $X_1(\xi_k)$ and $X_1(\xi'_k)$ tend to 0. We can say that this variable is a “free” pertinent variable. The role of this variable is to guarantee the fact that $F(\mathbb{C}^3)$ is dense in the target space $\mathbb{C}^3$.
2) F is of the form (**): Similarly to the case 1), we can see easily that F can admit only X_5 as a pertinent variable. Then the dimension of S is 1, which is a contradiction with the fact that the dimension of S is 2.

In conclusion, F has the following form:

$$\begin{array}{rcllll} F & = & \tilde{F}(X_1, & X_2, & X_3, & X_5) \\ \lim\limits_{k\to\infty} F(\xi_k) & = & \tilde{F}(0, & \lambda, & \lambda, & \mu) \\ \lim\limits_{k\to\infty} F(\xi'_k) & = & \tilde{F}(0, & \lambda', & \lambda', & \mu'), \end{array}$$

‘

Step 4: Describe the geometry of the 2-dimensional stratum S: On the one hand, the pertinent variables X_2 (or X_3) and X_5 tending independently to two complex numbers have degree 2; on the other hand, the degree of F is 2, then the degree of the surface S with respect to the variables λ and μ (or λ' and μ') is 1 (notice that by degree of S, we mean the degree of the equation defining S). We conclude that S is a plane.

In light of the example 3.5, we explicit now the algorithm for classifying the asymptotic sets of the non-proper dominant polynomial mappings $F = (F_1, F_2, F_3) : \mathbb{C}^3 \to \mathbb{C}^3$ of degree 2.

3.4. **Algorithm.**

Algorithm 3.10. We have the five following steps:

Step 1:

- Fix l *façons* $\kappa_1, \dots, \kappa_l$ in the list (3.4), where $1 \leqslant l \leqslant 19$.
- Determine the pertinent variables with respect to these l *façons* (knowing that they must be refered to a same mapping F).

Step 2:

- Assume that S is a 2-dimensional stratum of S_F admitting *only* the l *façons* $\kappa_1, \dots, \kappa_l$ in step 1.
- Take *generic* sequences $\xi_k^1, \dots, \xi_k^l$ corresponding to $\kappa_1, \dots, \kappa_l$, respectively.
- Compute the limit of the images of the sequences $\xi_k^1, \dots, \xi_k^l$ by the pertinent variables defined in step 1.
- Restrict the pertinent variables in step 1 using the fact $\dim S = 2$.

Step 3: Restrict again the pertinent variables in step 2 using the three following facts:

- the *façons* $\kappa_1, \dots, \kappa_l$ belongs to S: then the images of the generic sequences $\xi_k^1, \dots, \xi_k^l$ by the pertinent variables defined in the step 2 must tend to either an arbitrary complex number or zero,
- $\dim S = 2$: then there are at least two pertinent variables X_i and X_j such that the images of the sequences ξ_k and ξ'_k by X_i and X_j, respectively, tend independently to two complex numbers,
- F is dominant: then there are at least 3 independent pertinent variables (see lemma 2.2).

Step 4: Describe the geometry of the 2-dimensional irreductible stratum S of S_F in terms of the pertinent variables obtained in the step 3.

Step 5: Letting l run from 1 to 19 in the list (3.4).

Theorem 3.11. With the algorithm 3.10, we obtain the list of all possible asymptotic sets S_F of non-proper dominant polynomial mappings $F : \mathbb{C}^3 \to \mathbb{C}^3$ of degree 2.

Proof. On the one hand, the process of the algorithm 3.10 is possible, since the number of the *façons* in the list (3.4) is finite (19 *façons*). On the other hand, by the step 2, step 4 and step 5, we consider all the possible cases for all 2-dimensional irreductible strata of S_F. Since the dimension of S_F is 2 (see theorem 2.4), we get all the possible asymptotic sets S_F of non-proper dominant polynomial mappings $F : \mathbb{C}^3 \to \mathbb{C}^3$ of degree 2. ■

4. Results

In this section, we use the algorithm 3.10 to prove the following theorem.

Theorem 4.1. *The asymptotic set of a non-proper dominant polynomial mapping $F : \mathbb{C}^3 \to \mathbb{C}^3$ of degree 2 is one of the five elements in the following list $\mathcal{L}_{S_F}^{(3,2)}$. Moreover, any element of this list can be realized as the asymptotic set of a dominant polynomial mapping $F : \mathbb{C}^3 \to \mathbb{C}^3$ of degree 2.*

The list $\mathcal{L}_{S_F}^{(3,2)}$:

1) A plane.

2) A paraboloid.

3) The union of a plane $(\mathscr{P}) : \quad r_1x_1 + r_2x_2 + r_3x_3 + r_4 = 0$, and a plane of the form $(\mathscr{P}') :$ $r'_1x_1 + r'_2x_2 + r'_3x_3 + r'_4 = 0$, where we can choose two of the three coefficients r'_1, r'_2, r'_3, then the third of them and the fourth coefficient r'_4 are determined.

4) The union of a plane $(\mathscr{P}) : \quad r_1x_1 + r_2x_2 + r_3x_3 + r_4 = 0$ and a paraboloid of the form $(\mathscr{H}) : \quad r'_ix_i^2 + r'_jx_j + r'_lx_l + r'_4 = 0, \quad \{i, j, l\} = \{1, 2, 3\}$, where we can choose two of the three coefficients r'_1, r'_2, r'_3, then the third of them and the fourth coefficient r'_4 are determined.

5) The union of three planes

$$(\mathscr{P}) : \quad r_1x_1 + r_2x_2 + r_3x_3 + r_4 = 0,$$

$$(\mathscr{P}') : \quad r'_1x_1 + r'_2x_2 + r'_3x_3 + r'_4 = 0,$$

$$(\mathscr{P}'') : \quad r''_1x_1 + r''_2x_2 + r''_3x_3 + r''_4 = 0,$$

where:

a) for $(\mathscr{P}')$, we can choose two of the three coefficients r'_1, r'_2, r'_3, then the third of them and the fourth coefficient r'_4 are determined,

b) for $(\mathscr{P}'')$, we can choose two of the three coefficients r''_1, r''_2, r''_3, then the third of them and the fourth coefficient r''_4 are determined.

In order to prove this theorem, we need the two following lemmas.

Lemma 4.2. *Let $F = (F_1, F_2, F_3) : \mathbb{C}^3 \to \mathbb{C}^3$ be a non-proper dominant polynomial mapping of degree 2. If S_F contains a surface of degree higher than 1, then either S_F is a paraboloid, or S_F is the union of a paraboloid and a plane.*

Proof. Assume that S_F contains a surface $(\mathscr{H})$. Since $\deg F = 2$ then $1 \leqslant \deg(\mathscr{H}) \leqslant 2$.

A) We prove firstly that if S_F contains a surface $(\mathscr{H})$ where $\deg(\mathscr{H}) = 2$ then $(\mathscr{H})$ is a paraboloid. Since $\deg(\mathscr{H}) = 2$ and $\deg F = 2$ then S_F admits one *façon* κ in such a way that among the pertinent variables of F with respect to the *façon* κ, there exists only one *free* pertinent variable. That means, one of x_1, x_2 and x_3 is a pertinent variable of F with respect to κ (*cf.* Definition 3.1). Without loose of generality, we assume that x_1 is a pertinent variable of F with respect to the façon $\kappa = (3)[2]$. Assume that $\{\xi_k = (x_{1,k}, x_{2,k}, x_{3,k})\}$ is a generic sequence tending to infinity with the *façon* κ and

i) $x_{3,k}$ tends to infinity, $x_{2,k}$ tends to 0 in such a way that $x_{2,k}x_{3,k}$ tends to an arbitrary complex number λ,

ii) $x_{1,k}$ tends to an arbitrary complex number μ.

We see that $x_{1,k}^2$ and $(x_{1,k} + x_{2,k})x_{1,k}$ tend to μ^2. Since $\deg F = 2$ and $\deg(\mathscr{H}) = 2$, then

i) one coordinate polynomial F_η, where $\eta \in \{1, 2, 3\}$, must contain x_1 as an element of degree 1,

ii) the another coordinate polynomial $F_{\eta'}$, where $\eta' \in \{1, 2, 3\}$ and $\eta' \neq \eta$, must contain x_1^2 or $(x_1 + x_2)x_1$ as a pertinent variable.

Assume that the equation of the surface $(\mathscr{H})$ is $r_1\alpha_1^{p_1} + r_2\alpha_2^{p_2} + r_3\alpha_3^{p_3} + r_4 = 0$. Since $x_{1,k}^2$ and $(x_{1,k} + x_{2,k})x_{1,k}$ tend to the same complex number μ^2, and $\deg(\mathscr{H}) = 2$, then there exists an unique index $i \in \{1, 2, 3\}$ such that $r_i \neq 0$ and $p_i = 2$. If $r_j = 0$ or $p_j = 0$ for all $j \neq i$, $j \in \{1, 2, 3\}$, then $(\mathscr{H})$ is the union of two lines. That provides the contradiction with the fact that $\deg(\mathscr{H}) = 2$. So, there exists $j \neq i$, $j \in \{1, 2, 3\}$ such that $r_j \neq 0$ and $p_j = 1$. Consequently, the surface $(\mathscr{H})$ is a paraboloid.

B) We prove now that if S_F contains a paraboloid then the biggest possible S_F is the union of this paraboloid and a plane. Since S_F contains a paraboloid then with the same choice of the *façon* $\kappa = (3)[2]$ as in A), the mapping F must be considered as a dominant polynomial mapping of pertinent variables x_1, x_2, x_1x_2 and x_2x_3, that means:

$$F = \widetilde{F}(x_1, x_2, x_1x_2, x_2x_3).$$

We can see easily that if x_2 is a pertinent variable of F, then S_F admits only the *façon* κ and S_F is a paraboloid. Assume that S_F contains another irreductible surface $(\mathscr{H}')$ which is different from $(\mathscr{H})$. Then F must be considered as a polynomial mapping of pertinent variables x_1, x_1x_2 and x_2x_3, that means:

$$F = \widetilde{F}(x_1, x_1x_2, x_2x_3). \tag{4.3}$$

Let us consider now one *façon* κ' of $(\mathscr{H}')$ such that $\kappa' \neq \kappa$ and let $\{\xi'_k = (x'_{1,k}, x'_{2,k}, x'_{3,k})\}$ be a corresponding generic sequence of κ'. Notice that one coordinate of F admits x_1 as a pertinent variable. Let us show that $x'_{1,k}$ tends to 0. Assume that $x'_{1,k}$ tends to a non-zero complex number. As one coordinate of F admits x_1x_2 as a pertinent variable, then $x'_{2,k}$ does not tend to infinity. We have two cases:

+ If $x'_{2,k}$ tends to 0, then in order to ξ'_k tending to infinity, $x'_{3,k}$ must tend to infinity. Hence, the *façon* κ' is (3)[2]. That provides the contradiction with the fact $\kappa' \neq \kappa$.

+ If $x'_{2,k}$ tends to a non-zero finite complex number, since one coordinate of F admits x_2x_3 as factor, then $x'_{3,k}$ does not tend to infinity. That provides the contradiction with the fact that ξ'_k tends to infinity.

Therefore, $x'_{1,k}$ tends to 0. We have the following possible cases:

1) $\kappa' = (2)[1]$: then F is a polynomial mapping of the form $F = \widetilde{F}(x_1, x_3, x_1x_2, x_1x_3)$. Combining with (4.3), then $F = \widetilde{F}(x_1, x_1x_2)$. Therefore, F is not dominant, which provides the contradiction.

2) $\kappa' = (3)[1]$: then F is a polynomial mapping of the form $F = \widetilde{F}(x_1, x_2, x_1x_2, x_1x_3)$. Combining with (4.3), then $F = \widetilde{F}(x_1, x_1x_2)$. Therefore, F is not dominant, which provides the contradiction.

3) $\kappa' = (2,3)[1]$: then F is a polynomial mapping of the form $F = \widetilde{F}(x_1, x_1x_2, x_1x_3)$. Combining with (4.3), then $F = \widetilde{F}(x_1, x_1x_2)$. Therefore, F is not dominant, which provides the contradiction.

4) $\kappa' = (3)[1,2]$: then $F = \widetilde{F}(x_1, x_2, x_1x_2, x_2x_3, x_3x_1)$. Combining with (4.3), we have $F = \widetilde{F}(x_1, x_1x_2, x_2x_3)$. Since $x'_{1,k}x'_{2,k}$ and $x'_{1,k}$ tend to 0, then $\dim(\mathscr{H}') \leqslant 1$, that provides the contradiction.

5) $\kappa' = (2)[1,3]$: in this case, F is a polynomial mapping admitting the form

$$F = \widetilde{F}(x_1, x_3, x_1x_2, x_2x_3, x_3x_1).$$

Combining with (4.3), then

$$F = \widetilde{F}(x_1, x_1x_2, x_2x_3).$$

We know that $x'_{1,k}$ tends to 0. Assume that $x'_{1,k}x'_{2,k}$ tends to a complex number λ and $x'_{2,k}x'_{3,k}$ tends to a complex number μ, we have

$$(\mathscr{H}') = \{(\widetilde{F}_1(0,\lambda,\mu), \widetilde{F}_2(0,\lambda,\mu), \widetilde{F}_3(0,\lambda,\mu)) : \lambda, \mu \in \mathbb{C}\},$$

where $\widetilde{F} = (\widetilde{F}_1, \widetilde{F}_2, \widetilde{F}_3)$. Since $\deg F = 2$, then the degree of $\widetilde{F}_i$ with respect to the variables λ and μ must be 1, for all $i \in \{1,2,3\}$. Consequently, the surface $(\mathscr{H}')$ is a plane.

$\blacksquare$

Lemma 4.4. *Let $F : \mathbb{C}^3 \to \mathbb{C}^3$ be a non-proper dominant polynomial mapping of degree 2. Assume that S is a 2-dimensional irreductible stratum of S_F. Then S admits at most two façons. Moreover, if S admits two façons, then S_F is a plane.*

Proof. Let $F : \mathbb{C}^3 \to \mathbb{C}^3$ be a non-proper dominant polynomial mapping of degree 2. Assume that S is a 2-dimensional irreductible stratum of S_F.

A) We provide firstly the list of pairs of *façons* that S can admit and we write F in terms of pertinent variables in each of these cases. Let us fix a pair of *façons* (κ, κ') in the list (3.4) and assume that S admits these two *façons*. We use the steps 1, 2, 3 and 4 of the algorithm 3.10. In the same way than the example 3.5, we can determine the form of F in terms of its pertinent variables with respect to two fixed *façons* after using the conditions of dimension of S and the dominancy of F. Letting two *façons* κ, κ' run in the list (3.4), we get the following possiblilities:

1) $(\kappa, \kappa') = ((1,2,3), (i_1, i_2)[j])$, where $\{i_1, i_2, j\} = \{1, 2, 3\}$ and

$$F = \tilde{F}(x_{i_1} - x_{i_2}, (x_{i_1} - x_{i_2})x_{i_1}, (x_{i_1} - x_{i_2})x_{i_2}, (x_{i_1} - x_{i_2})x_j, (x_{i_1} - x_j)x_j).$$

2) $(\kappa, \kappa') = ((1,2,3), (i)[j_1, j_2])$, where $\{i, j_1, j_2\} = \{1, 2, 3\}$ and

$$F = \tilde{F}((x_i - x_{j_1})x_{j_1}, (x_i - x_{j_1})x_{j_2}, (x_i - x_{j_2})x_{j_1}, (x_i - x_{j_2})x_{j_2}, (x_{j_1} - x_{j_2}), (x_{j_1} - x_{j_2})x_i).$$

3) $(\kappa, \kappa') = ((1,2)[3], (i)[3, j])$, where $\{i, j\} = \{1, 2\}$, and

$$F = \tilde{F}(x_3, x_j x_3, x_i x_3, (x_i - x_j)x_j).$$

4) $(\kappa, \kappa') = ((1,2)[3], (3)[1,2])$ and

$$F = \tilde{F}(x_1x_3, x_2x_3, x_1 - x_2, r_1x_1x_3 + r_2x_2x_3 + r_3(x_1 - x_2)x_1 + r_4(x_1 - x_2)x_2),$$

where $r_l \in \mathbb{C}$, for $l = 1, \ldots, 4$, such that $(r_1 \neq 0, r_2 \neq 0, (r_3, r_4) \neq (0,0))$, or $((r_1, r_2) \neq (0,0), (r_3, r_4) \neq (0,0))$.

5) $(\kappa, \kappa') = ((1,3)[2], (i)[2, j])$, where $\{i, j\} = \{1, 3\}$, and

$$F = \tilde{F}(x_2, x_2x_j, x_ix_2, (x_i - x_j)x_j).$$

6) $(\kappa, \kappa') = ((1,3)[2], (2)[1,3])$ and

$$F = \tilde{F}(x_1x_2, x_2x_3, x_1 - x_3, r_1x_1x_2 + r_2x_2x_3 + r_3(x_1 - x_3)x_1 + r_4(x_1 - x_3)x_3),$$

where $r_l \in \mathbb{C}$, for $l = 1, \ldots, 4$, such that $(r_1 \neq 0, r_2 \neq 0)$, or $(r_1, r_2) \neq (0,0)$ and $(r_3, r_4) \neq (0,0)$.

7) $(\kappa, \kappa') = ((2,3)[1], (i)[1, j])$, where $\{i, j\} = \{2, 3\}$, and

$$F = \tilde{F}(x_1, x_1x_i, x_1x_j, (x_i - x_j)x_j).$$

8) $(\kappa, \kappa') = ((2,3)[1], (1)[2,3])$ and

$$F = \tilde{F}(x_1x_3, x_1x_2, x_3 - x_2, r_1x_1x_3 + r_2x_1x_2 + r_3(x_3 - x_2)x_3 + r_4(x_3 - x_2)x_2),$$

where $r_l \in \mathbb{C}$, for $l = 1, \dots, 4$, such that $(r_1 \neq 0, r_2 \neq 0, (r_3, r_4) \neq (0,0))$ or $((r_1, r_2) \neq (0,0), (r_3, r_4) \neq (0,0))$.

9) $(\kappa,\, \kappa') = ((1)[2,3],\, (i)[1,j])$, where $\{i, j\} = \{2, 3\}$, and

$$F = \tilde{F}(x_j, x_1x_i, r_1x_ix_j + r_2x_1x_j),$$

where r_1 et r_2 are the non-zero complex numbers.

B) We prove now that S admits at most two *façons*. We prove the result for the first case of the above possibilities: $(\kappa,\, \kappa') = ((1,2,3), (i_1, i_2)[j])$, where $\{i_1, i_2, j\} = \{1, 2, 3\}$. The other cases are proved similarly. For example, assume that S admits two *façons* $\kappa = (1,2,3)$ and $\kappa' = (1,2)[3]$. We prove that S cannot admit the third *façon* κ'' different from κ and κ'.

Let κ'' be a *façon* of S_F. Let us denote by $\{\xi''_k = (x''_{1,k}, x''_{2,k}, x''_{3,k})\}$ a generic sequence corresponding to κ''. By the example 3.5, the mapping F admits X_1, X_2 (or X_3), and X_5 as the pertinent variables, where

$$X_1 = x_1 - x_2, \quad X_2 = (x_1 - x_2)x_1, \quad X_3 = (x_1 - x_2)x_2, \quad X_5 = (x_1 - x_3)x_3.$$

Without loose the generality, we can assume that X_2 is a pertinent variable of F. We prove that $X_1(\xi''_k) = (x''_{1,k} - x''_{2,k})$ tends to 0. Assume that $X_1(\xi''_k) = (x''_{1,k} - x''_{2,k})$ tends to a non-zero complex number. Then :

+ If $x''_{1,k}$ tends to infinity, then $X_2(\xi''_k)$ tends to infinity, that provides a contradiction with the fact that X_2 is a pertinent variable of F.
+ If $x''_{2,k}$ tends to infinity, then $x''_{1,k}$ also tends to infinity since $(x''_{1,k} - x''_{2,k})$ tends to a non-zero complex number. That implies $X_2(\xi''_k)$ tends to infinity and this provides a contradiction with the fact that X_2 is a pertinent variable of F.

Hence, $x''_{1,k}$ and $x''_{2,k}$ cannot tend to infinity. Consequently, $x''_{3,k}$ must tend to infinity. Therefore, $X_5(\xi''_k) = (x''_{1,k} - x''_{3,k})x''_{3,k}$ tends to infinity, that provides the contradiction with the fact that X_5 is a pertinent variable of F. We conclude that $(x''_{1,k} - x''_{2,k})$ tend to 0.

Then we have two possibilities:

a) either both of $x''_{1,k}$ and $x''_{2,k}$ tend to 0: then $X_1(\xi''_k) = (x''_{1,k} - x''_{2,k})$, $X_2(\xi''_k) = (x''_{1,k} - x''_{2,k})x''_{1,k}$ and $X_3(\xi''_k) = (x''_{1,k} - x''_{2,k})x''_{2,k}$ tend to 0, which provides the contradiction with the fact that the dimension of S is 2,

b) or both of $x''_{1,k}$ and $x''_{2,k}$ tend to infinity: Since X_5 is a pertinent variable of F, then $x''_{3,k}$ tends to 0 or infinity. We conclude that the *façon* κ'' is $(1,2,3)$ or $(1,2)[3]$.

In conclusion, S admits only the two *façons* $\kappa = (1,2,3)$ and $\kappa' = (1,2)[3]$.

C) We prove now that if there exists a 2-dimensional irreductible stratum S of S_F admitting two *façons*, then S_F is a plane. Similarly to B), we prove this fact for the first case of the possibilities in A), that means, the case of $(\kappa,\, \kappa') = ((1,2,3), (i_1, i_2)[j])$, where $\{i_1, i_2, j\} = \{1, 2, 3\}$. The other cases are proved similarly. For example, assume that S admits two *façons*

$\kappa = (1,2,3)$ and $\kappa' = (1,2)[3]$. With the same arguments than in the example 3.5, the stratum S is a plane. By B), the asymptotic set S_F admits also *only* two *façons* $\kappa = (1,2,3)$ and $\kappa' = (1,2)[3]$. In other words, S_F and S concide. We conclude that S_F is a plane.

■

We prove now the theorem 4.1.

Proof. (*The proof of theorem 4.1*). The cases 1) and 2) are easily achievable by the lemmas 4.4 and 4.2, respectively. Let us prove the cases 3), 4) and 5). In these cases, on the one hand, since S_F contains at least two irreductible surfaces, then S_F admits at least two *façons*; on the other hand, by the lemma 4.4, each irreductible surface of S_F admits only one *façon*. Assume that κ, κ' are two different *façons* of S_F and $\{\xi_k\}$, $\{\xi'_k\}$ are two corresponding generic sequences, respectively. We use the algorithm 3.10 and in the same way than the proofs of the lemmas 4.2 and 4.4, we can see easily that the pairs of *façons* (κ, κ') must belong to only the following pairs of groups: (I, IV), (I, V), (I, VI), (II, VI), (IV, V), (IV, VI), (V, VI) and (VI, VI) in the list (3.4).

i) If κ belongs to the group I and κ' belongs to the group IV, for example $\kappa = (1,2,3)$ and $\kappa' = (1,2)[3]$. From the example 3.5, F is a dominant polynomial mapping which can be written in terms of pertinent variables:

$$\begin{array}{rclllll} F & = & \tilde{F}(x_1 - x_2, & (x_1 - x_2)x_1, & (x_1 - x_2)x_2, & (x_1 - x_2)x_3, & (x_1 - x_3)x_3) \\ \lim\limits_{k\to\infty} F(\xi_k) & = & \tilde{F}(0, & \lambda, & \lambda, & \lambda, & \mu), \\ \lim\limits_{k\to\infty} F(\xi'_k) & = & \tilde{F}(0, & \lambda', & \lambda', & 0, & \mu'), \end{array}$$

where $\lambda, \mu, \lambda', \mu' \in \mathbb{C}$ (see (3.6), (3.7) and (3.8)). We see that, with the sequence $\{\xi_k\}$, the pertinent variables tending to an arbitrary complex numbers have the degree 2, then the *façon* κ provides a plane, since the degree of F is 2. In the same way, the *façon* κ' provides a plane. Furthermore, it is easy to check that these two planes must have the form of the case 3) of the theorem and S_F is the union of these two planes.

ii) If κ belongs to the group I and κ' belongs to the group V, for example $\kappa = (1,2,3)$ and $\kappa' = (1)[2]$. then, on the one hand, F is a dominant polynomial mapping which can be written in terms of pertinent variables:

$$F = \tilde{F}(x_2 - x_3, (x_2 - x_3)x_2, (x_2 - x_3)x_3, (x_1 - x_2)x_2).$$

On the other hand, with the same arguments than the example 3.5, and for suitable generic sequences $\{\xi_k\}$ and $\{\xi'_k\}$, we obtain:

$$\begin{array}{rclllll} \lim\limits_{k\to\infty} F(\xi_k) & = & \tilde{F}(0, & \lambda, & \lambda, & \mu), \\ \lim\limits_{k\to\infty} F(\xi'_k) & = & \tilde{F}(\lambda', & 0, & \lambda'^2, & \mu'), \end{array}$$

where $\lambda, \mu, \lambda', \mu' \in \mathbb{C}$. With the same arguments than in the case i), we have:

a) either the *façons* κ and κ' provide two planes of the form of the case 3) of our theorem,

b) or the *façon* κ provides the plane $(\mathscr{P})$ and, by the lemma 4.2, the *façon* κ' provides the paraboloid $(\mathscr{H})$ of the form of the case 4) of our theorem.

By an easy calculation, we see that if S_F admits another *façon* κ'' which is different from the *façons* κ and κ', then this *façon* provides a 1-dimensional stratum contained in $(\mathscr{P})$ or contained in $(\mathscr{H})$.

iii) Proceeding in the same way for the cases where (κ, κ') is a pair of *façons* belonging to the pairs of groups: (I, VI), (II, VI), (IV, V), (IV, VI) and (V, VI), we obtain the case 3) or the case 4) of the theorem.

iv) Consider now the case where κ and κ' belong to the group VI, for example, $\kappa = (1)[2,3]$ and $\kappa' = (2)[1,3]$, then F is a dominant polynomial mapping which can be written in terms of pertinent variables:

$$F = \tilde{F}(x_3, x_1x_2, x_2x_3, x_3x_1).$$

With the same arguments than the example 3.5, and for suitable generic sequences $\{\xi_k\}$ and $\{\xi'_k\}$, we obtain:

$$\begin{array}{rcl} \lim\limits_{k\to\infty} F(\xi_k) & = & \tilde{F}(0, \quad \lambda, \quad 0, \quad \mu), \\ \lim\limits_{k\to\infty} F(\xi'_k) & = & \tilde{F}(0, \quad \lambda', \quad \mu', \quad 0), \end{array}$$

where $\lambda, \mu, \lambda', \mu' \in \mathbb{C}$. In this case, we have two possibilities:

a) either F admits x_3 as a pertinent variable: This case is similar to the case i) and we have the case 3) of the theorem,

b) or F does not admit x_3 as a pertinent variable, that means

$$F = \tilde{F}(x_1x_2, x_2x_3, x_3x_1). \tag{4.4}$$

In this case, S_F admits one more *façon* $\kappa'' = (3)[1,2]$ such that with a corresponding suitable generic sequence $\{\xi''_k\}$ of κ'', we have

$$\begin{array}{rcl} \lim\limits_{k\to\infty} F(\xi_k) & = & \tilde{F}(\lambda, \quad 0, \quad \mu), \\ \lim\limits_{k\to\infty} F(\xi'_k) & = & \tilde{F}(\lambda', \quad \mu', \quad 0), \\ \lim\limits_{k\to\infty} F(\xi''_k) & = & \tilde{F}(0, \quad \lambda'', \quad \mu''), \end{array}$$

where $\lambda'', \mu'' \in \mathbb{C}$. In this case, S_F is the union of three planes the forms of which are as in the case 5) of the theorem. ■

5. The general case

The algorithm 3.10 can be generalized to clasify the asymptotic sets of non-proper dominant polynomial mappings $F : \mathbb{C}^n \to \mathbb{C}^n$ of degree d where $n \geqslant 3$ and $d \geqslant 2$ as the following.

Algorithm 5.1. We have the six following steps:

Step 1: Determine the list $\mathcal{L}_F^{(n,d)}$ of all the possible *façons* of S_F.

Step 2: Fix l *façons* $\kappa_1, \ldots, \kappa_l$ in the list $\mathcal{L}_F^{(n,d)}$ obtained in step 1. Determine the pertinent variables with respect to these l *façons* (in the similar way than the definition 3.1).

Step 3:

- Assume that S is a $(n-1)$-dimensional stratum of S_F admiting *only* the l *façons* $\kappa_1, \ldots, \kappa_l$ determined in step 1.
- Take *generic* sequences $\xi_k^1, \ldots, \xi_k^l$ corresponding to $\kappa_1, \ldots, \kappa_l$, respectively.
- Compute the limit of the images of the sequences $\xi_k^1, \ldots, \xi_k^l$ by pertinent variables defined in step 1.
- Restrict the pertinent variables defined in step 2 using the fact $\dim S = n-1$.

Step 4: Restrict again the pertinent variables in step 3 using the three following facts:

- all the *façons* $\kappa_1, \ldots, \kappa_l$ belong to S: then the images of the generic sequences $\xi_k^1, \ldots, \xi_k^l$ by the pertinent variables defined in the step 2 must tend to either an arbitrary complex number or zero,
- $\dim S = n-1$: then there are at least $n-1$ pertinent variables $X_{i_1}, \ldots, X_{i_{n-1}}$ such that the images of the sequences $\{\xi_k^1\}, \ldots, \{\xi_k^l\}$ by $X_{i_1}, \ldots, X_{i_{n-1}}$, respectively, tend independently to $(n-1)$ complex numbers,
- F is dominant: then there are at least n independent pertinent variables (see lemma 2.2).

Step 5: Describe the geometry of the $(n-1)$-dimensional irreductible stratum S in terms of the pertinent variables obtained in the step 4.

Step 6: Let l run in the list obtained in the step 1.

Theorem 5.2. With the algorithm 5.1, we obtain all possible asymptotic sets of non-proper dominant polynomial mapping $F : \mathbb{C}^n \to \mathbb{C}^n$ of degree d.

Proof. In the one hand, by theorem 2.4, the dimension of S_F is $n-1$. By the step 3, step 5 and step 6, we consider all the possible cases of the all $(n-1)$-dimensional irreductible strata of S_F. Since the dimension of S_F is $n-1$ (see theorem 2.4), we get all the possible asymptotic sets S_F of non-proper dominant polynomial mappings $F : \mathbb{C}^n \to \mathbb{C}^n$ of degree d. In the other hand, the number of all the possible *façons* of a polynomial mapping $F : \mathbb{C}^n \to \mathbb{C}^n$ is finite, as the shown of the following lemma: ■

Lemma 5.3. Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial mapping such that $S_F \neq \emptyset$. Then, the number of all possible *façons* of S_F is finite. More precisely, the maximum number of *façons* of S_F is equal to

$$\sum_{t=1}^{n} C_t^n + \sum_{t=1}^{n-1} C_t^n + \sum_{t=2}^{n-1} A_t^n,$$

where

$$C_t^n = \frac{n!}{t!(n-t)!}, \qquad A_t^n = \frac{n!}{(n-t)!}.$$

Proof. Assume that $\kappa = (i_1, \dots, i_p)[j_1, \dots, j_q]$ is a *façon* of S_F. We have the following cases:

i) If $\{i_1, \dots, i_p\} \cup \{j_1, \dots, j_q\} = \{1, \dots, n\}$: we have $\sum_{t=1}^{n} C_t^n$ possible *façons.*

ii) If $\{i_1, \dots, i_p\} \cup \{j_1, \dots, j_q\} \neq \{1, \dots, n\}$ and $\{j_1, \dots, j_q\} = \varnothing$: we have $\sum_{t=1}^{n-1} C_t^n$ possible *façons.*

iii) If $\{i_1, \dots, i_p\} \cup \{j_1, \dots, j_q\} \neq \{1, \dots, n\}$ and $\{j_1, \dots, j_q\} \neq \varnothing$: We have $\sum_{t=2}^{n-1} A_t^n$ possible *façons.*

As the three cases are independent, then the maximum number of *façons* of S_F is equal to

$$\sum_{t=1}^{n} C_t^n + \sum_{t=1}^{n-1} C_t^n + \sum_{t=2}^{n-1} A_t^n.$$

■

Remark 5.4. In the example 3.5 and in the proofs of the lemmas 4.2 and 4.4, we use a linear change of variables to simplify the pertinent variables (so that we can work without coefficients and then we can simplify calculations). This change of variables does not modify the results of the theorem 4.1. However, in the algorithms 3.10 and 5.1, we do not need the step of linear change of variables, since the computers can work with coefficients of pertinent variables without making the problem heavier.

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