Singularities with symmetry

School on Singularity Theory
Mini-course on Applications of Singularities

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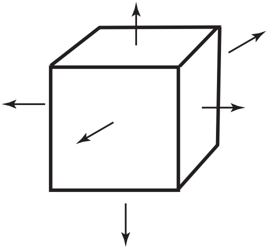
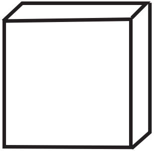
Deformation of an elastic cube under uniform traction

Dimensions of the deformed cube: (ℓ_1, ℓ_2, ℓ_3) $\ell_j > 0$

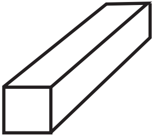
Conservation of volume: $\ell_1 \ell_2 \ell_3 = 1$ Traction strength: λ



Symmetric deformations



$\ell_1 < \ell_2 = \ell_3$
plate



$\ell_1 = \ell_2 < \ell_3$
rod

Symmetries of the problem

Permutations of the sides symmetry group $\mathbf{S}_3$.

Model independent treatment

General method, applied to the problem of the cube:

Bifurcation, reduction to kernel

Symmetry, multidimensional kernel

Fixed-point subspaces

Invariant theory

Equivariant singularities

Bifurcation

Model is an equation $\dot{x} = g(x, \lambda)$ $x \in \mathbf{R}^n$.
Equilibrium solutions satisfy $g(x_0, \lambda_0) = 0$.
Stability is given by the sign of the eigenvalues of $D_x g(x_0, \lambda_0)$.

If all eigenvalues are non-zero, then, locally,
$\exists! x(\lambda) \quad x(\lambda_0) = x_0$ such that $g(x(\lambda), \lambda) \equiv 0$
and these are all the solutions of $g(x, \lambda) = 0$.
(Implicit function theorem)

Without loss of generality, suppose $g(0, \lambda) = 0$.

Bifurcation

Model is an equation $\dot{x} = g(x, \lambda)$ $x \in \mathbf{R}^n$, $g(0, \lambda) = 0$.

Bifurcation at $\lambda = 0$ implies zero is eigenvalue of $D_x g(0, 0)$

$\ker D_x g(0, 0) \neq \{0\}$

Liapunov-Schmidt reduction (Implicit function theorem)
problem is equivalent to

$$\dot{x} = h(x, \lambda) \qquad x \in \ker D_x g(0, 0) \qquad h(0, \lambda) = 0$$

Generically, $\dim \ker D_x g(0, 0) = 1$.

Symmetry

Model is an equation $\dot{x} = g(x, \lambda)$ $x \in \mathbf{R}^n$.

$\gamma \in \mathbf{O}(n)$ is a symmetry of $x_0 \in \mathbf{R}^n$

$$\gamma x_0 = x_0$$

$\gamma \in \mathbf{O}(n)$ is a symmetry of g

$$g(\gamma x, \lambda) = \gamma g(x, \lambda)$$

(g is γ-equivariant)

If $\gamma x_0 = x_0$ and $D_x g(x_0, \lambda_0) = L$ then $\gamma L = L\gamma$.

If $Lv = 0$ then $L\gamma v = 0$.

Generically, in systems with symmetry, $\dim \ker D_x g(x_0, \lambda_0) > 1$.

Symmetry

Example: symmetries of an equilateral triangle

$\mathbf{D}_3$ acting on $\mathbf{R}^2 \sim \mathbf{C}$ by

$$\rho(z) = e^{2\pi i/3} z \qquad \kappa(z) = \bar{z}$$

κ is a symmetry of $x \in \mathbf{R} \subset \mathbf{C}$.
If $g(x, \lambda)$ is $\mathbf{D}_3$-equivariant with $D_x g(0,0) = L$
and if $Lv = 0$ with $v \neq 0$ then $L\rho v = 0$.
Since $v \neq 0$ and ρv are linearly independent, then $L \equiv 0$,
$\dim \ker D_x g(x_0, \lambda_0) = 2$.

Fixed-point subspaces— spontaneous symmetry breaking

$\Gamma \leq \mathbf{O}(n)$ group of symmetries γ of bifurcation problem $g(x, \lambda)$.
$\Sigma \leq \Gamma$ subgroup of symmetries of an equilibrium $x_0 \in \mathbf{R}^n$

isotropy subgroup

All the equilibria with symmetry subgroup Σ lie in the:

Fixed-point subspace

$\mathrm{Fix}(\Sigma) = \{x \in \mathbf{R}^n : \ \gamma x = x \quad \forall \gamma \in \Sigma\}$
If $x \in \mathrm{Fix}(\Sigma)$ then $g(x, \lambda) \in \mathrm{Fix}(\Sigma)$.

Action of Γ on V is irreducible

if the only subspaces $W \leq V$

such that $\gamma w \in W \ \forall w \in W, \gamma \in \Gamma$

are $W = \{0\}$ and $W = V$.

Theorem (Equivariant branching lemma)

If the action of Γ on V is irreducible, and if $\Sigma \leq \Gamma$ is an isotropy subgroup with $\dim \mathrm{Fix}(\Sigma) = 1$, then, generically, for a Γ-equivariant bifurcation problem $g(x, \lambda)$, there is a unique branch with symmetry Σ that bifurcates from the trivial equilibrium at $\lambda = 0$.

Fixed-point subspaces and the cube

Symmetry group $\Gamma = \mathbf{S}_3$
acting on the surface $\{(\ell_1, \ell_2, \ell_3) : \ \ell_j > 0 \quad \ell_1\ell_2\ell_3 = 1\} \subset \mathbf{R}^3$

$\mathbf{S}_3$ is generated by:

$$P_3(\ell_1, \ell_2, \ell_3) = (\ell_2, \ell_3, \ell_1) \qquad P_2(\ell_1, \ell_2, \ell_3) = (\ell_2, \ell_1, \ell_3)$$

action of $\mathbf{S}_3$ on the surface is isomorphic to $\mathbf{D}_3$ acting on $\mathbf{R}^2 \sim \mathbf{C}$

$$\rho(z) = e^{2\pi i/3} z \approx P_3 \qquad \kappa(z) = \bar{z} \approx P_2$$

cube $\ell_1 = \ell_2 = \ell_3$ identified to origin (maximum symmetry)

$\dim \mathrm{Fix}(\Sigma) = 1$ for $\Sigma = \mathbf{Z}_2 = \{\kappa, Id\}$.

A branch with symmetry κ : rods or plates.

Three symmetric copies of the branch.

are they rods or are they plates?

Invariant functions

$f : \mathbf{R}^n \longrightarrow \mathbf{R}$ is Γ-invariant if

$$f(\gamma x) = f(x) \qquad \forall x \in \mathbf{R}^n \qquad \forall \gamma \in \Gamma$$

Example: $\mathbf{D}_3$-invariant functions

$$u(x,y) = x^2 + y^2 \qquad\qquad v(x,y) = x^3 - 3xy^2$$

in complex notation

$$u(z) = z\bar{z} \qquad\qquad v(z) = \mathrm{Re}(z^3)$$

Every $\mathbf{D}_3$-invariant C^∞ function can be written in the form

$$f(x,y) = p(u(x,y), v(x,y))$$

where p is a C^∞ function of 2 variables.

(by the Hilbert-Weyl-Schwarz theorem)

Equivariant vector fields

$F : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ is Γ-equivariant if

$$F(\gamma x) = \gamma F(x) \qquad \forall x \in \mathbf{R}^n \qquad \forall \gamma \in \Gamma$$

If $f : \mathbf{R}^n \longrightarrow \mathbf{R}$ is Γ-invariant
and if $F : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ is Γ-equivariant
then $f(x)F(x)$ is also Γ-equivariant.

Equivariant vector fields

Example: $\mathbf{D}_3$-equivariant vector fields

$$X(x,y) = (x,y) \qquad\qquad Y(x,y) = (x^2 - y^2, -2xy)$$

in complex notation

$$X(z) = z \qquad\qquad Y(z) = (\bar{z})^2$$

Every $\mathbf{D}_3$-equivariant vector field of class C^∞ can be written in the form

$$F(x,y) = p(u(x,y), v(x,y))X(x,y) + q(u(x,y), v(x,y))Y(x,y)$$

where p and q are C^∞ functions of 2 variables.

(by Poénaru's theorem)

$\mathbf{D}_3$-equivariant vector fields generated by:

Invariant functions

$$u(x,y) = x^2 + y^2 \qquad v(x,y) = x^3 - 3xy^2$$

Equivariant vector fields

$$X(x,y) = (x,y) \qquad Y(x,y) = (x^2 - y^2, -2xy)$$

$\mathbf{D}_3$-equivariant Taylor expansions:

degree	term
1	X
2	Y
3	uX
4	$uY \quad vX$
5	$u^2X \quad vY$

Consequence

Theorem

The rod and plate solutions that bifurcate from the cube are unstable.

Proof

Model is an equation $(\dot{x}, \dot{y}) = g(x, y, \lambda)$
$(x, y) \in \mathbf{R}^2$ with g $\mathbf{D}_3$-equivariant.

$$g(x, y, \lambda) = p(u, v, \lambda) \begin{pmatrix} x \\ y \end{pmatrix} + q(u, v, \lambda) \begin{pmatrix} x^2 - y^2 \\ -2xy \end{pmatrix}$$

Bifurcating solutions are in $\mathrm{Fix}(\mathbf{Z}_2) = \{(x, 0)\}$, hence they satisfy

$$xp + x^2 q = 0$$

and if $x \neq 0$ then

$$p + xq = 0$$

Proof of instability

Write $p_x = \dfrac{\partial p}{\partial x} = 2x\dfrac{\partial p}{\partial u} + 3(x^2 - y^2)\dfrac{\partial p}{\partial v}$ and similarly for q_x, p_y, q_y.

Then $D_{(x,y)}g(x, y, \lambda)$ is:

$$\begin{pmatrix} xp_x + p + (x^2 - y^2)q_x + 2xq & xp_y + (x^2 - y^2)q_y - 2yq \\ yp_x - 2xyq_x - 2yq & ypy + p - 2xyq_y - 2xq \end{pmatrix}$$

In $\mathrm{Fix}(\mathbf{Z}_2) = \{(x, 0, \lambda)\}$, $D_{(x,y)}g(x, 0, \lambda)$ is:

$$\begin{pmatrix} p + 2xq + xp_x + x^2 q_x & xp_y + x^2 q_y \\ 0 & p - 2xq \end{pmatrix}$$

Eigenvalues:

$p + 2xq + xp_x + x^2 q_x = p + 2xq + O(x^3)$ $\qquad$ $p - 2xq$

Proof of instability

Eigenvalues of $D_{(x,y)}g(x,0,\lambda)$:

$p + 2xq + O(x^3)$ $\qquad$ $p - 2xq$

In $\mathrm{Fix}(\mathbf{Z}_2) = \{(x,0)\}$ solutions of $g = 0$ satisfy

$$p + xq = 0$$

Hence the eigenvalues are

$p + 2xq + O(x^3) = xq + O(x^3)$ $\qquad$ $p - 2xq = -3xq$

that have opposite signs for small $x \neq 0$ — the branch is unstable.

Equivariant singularities

How to use singularity theory to find stable branches of $g(x, y, \lambda) = pX + qY$:

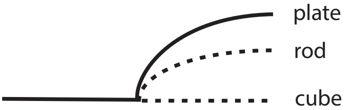
- Find a more degenerate problem nearby (organising centre)
  Suppose $p(0, 0, 0) = 0$ and also $q(0, 0, 0) = 0$.
- Perturb the degenerate problem generically (within the class of $\mathbf{D}_3$-equivariant bifurcations) and see what happens.

If $p(0, 0, 0) = 0 = q(0, 0, 0)$, then, up to changes of coordinates that preserve the $\mathbf{D}_3$ symmetry, g is like:
$h(z, \lambda) = (u - \lambda)z + (u + mv)\bar{z}^2$

Bifurcation diagrams

Organising centre

$$h(z, \lambda) = (u - \lambda)z + (u + mv)\bar{z}^2$$



solid line — stable

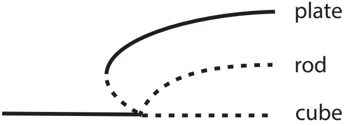
dashed line — unstable

Bifurcation diagrams

Generic perturbation

$h(z, \lambda) = (u - \lambda)z + (u + mv + \alpha)\bar{z}^2$

$\alpha > 0$

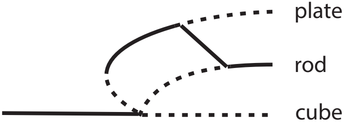


solid line — stable

dashed line — unstable

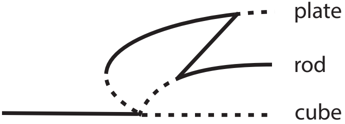
$\alpha < 0$

$m > 0$



$\alpha < 0$

$m < 0$



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