Classification of multigerms (from a modern viewpoint)

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Abstract

The goal of these notes is to give an overview of the state of the art in classification of multigerms. We have tried to make them self-contained but certainly not extensive. The results included here scope most of the research on classification of multigerms carried out in the last 15 years with special emphasis on recent publications and preprints written by the authors of these notes and their collaborators.

Chapter 1

Introduction

The foundations of Singularity Theory of Differentiable Maps can be considered to be the fundamental works by Whitney, Thom and Mather. Their main concern was the classification of singularities of map germs from $\mathbb{K}^n$ to $\mathbb{K}^p$, for example, Whitney classified stable maps from the plane into the plane in [36], proving that any stable germ is equivalent to the fold or the cusp. Thom's work on Catastrophe Theory and Mather's work ([14, 15, 16]) was followed by Arnold's classification of simple singularities of functions in [2]. Since then this has been one of the main areas of research in Singularity Theory. In fact, complete classifications up to certain codimension for certain pairs (n, p) have been carried out by many authors ([30], [31], [19], [7], [17], [34], ...) and it is still an active field of research.

The bibliography related to the classification of multigerms is less abundant. The first reference is Mather's classification of stable multigerms [16]. Goryunov gave in [8] a list of multigerms without normal forms from $\mathbb{R}^2$ to $\mathbb{R}^3$ including codimension 1 singularities. Hobbs and Kirk ([9]) and the second author in [34] give a classification of all simple multigerms for this case. In fact, in [34] a method that can be applied to the case $\mathbb{C}^n$ to $\mathbb{C}^{n+1}$ is given. Normal forms for multigerms from the plane to the plane are given by several authors. A good account of this is [23]. Kolgushkin and Sadykov studied simple multigerms of curves in [13]. The first author and Romero Fuster give normal forms of multigerms up to codimension 2 from $\mathbb{R}^3$ to $\mathbb{R}^3$ in [25].

The classification of multigerms using the classical Singularity Theory techniques can be extremely hard to deal with. On the other hand, the need of new classifications, including multigerms, with larger (n, p) is growing due to its applications to related areas in Singularity Theory such as topological invariants or generic geometry. Therefore new techniques have been developed. For example, operations to obtain germs in certain dimensions from germs with fewer branches in lower dimensions have proved to be a very important tool.

In the present notes we collect the results and new techniques which have successfully classified all corank 1 codimension 1 ([6]) and 2 ([26]) multigerms, amongst other things. Chapter 2 comprises the minimum necessary previous knowledge and notation in order to follow the notes. Most results in this Chapter are due to Mather and can be found in more detail in the survey article by Wall ([33]) or in the draft of the future book by Mond and Nuño-Ballesteros ([20]). Chapter 3 introduces some new techniques and operations necessary to obtain all codimension 1 multigerms. Chapter 4 generalizes the operations introduced in Chapter 3 and shows how to construct all codimension 2 multigerms for any pair of dimensions in the nice dimensions. Chapter 5 is a brief account on the simplicity of multigerms obtained via the techniques described in the notes. Finally, Chapter 6 gives some ideas for future research.

Chapter 2

Preliminaries

We include here the necessary preliminaries to be able to follow the following chapters. For a complete introduction to Singularity Theory of Differentiable Maps we refer to the draft of the future book by Nuno-Ballesteros and Mond [20] or to the survey by C.T.C. Wall [33].

Let $\mathcal{O}_n^p$ be the vector space of monogerms with n variables and p components, i.e. whose elements are germs of mappings $f : (\mathbb{K}^n, x_0) \to (\mathbb{K}^p, y_0)$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. When $p = 1$, $\mathcal{O}_n^1 = \mathcal{O}_n$ is the local ring of germs of functions in n-variables and $\mathcal{M}_n$ its maximal ideal. The set $\mathcal{O}_n^p$ is a free $\mathcal{O}_n$-module of rank p. A multigerm is a germ of an analytic (complex case) or smooth (real case) map $f = \{f_1, \ldots, f_r\} : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ where $S = \{x_1, \ldots, x_r\} \subset \mathbb{K}^n$, $f_i : (\mathbb{K}^n, x_i) \to (\mathbb{K}^p, 0)$. Let $\mathcal{M}_n\mathcal{O}_{n,S}^p$ be the vector space of such map germs. We call f_i, $i = 1, \ldots, r$, a branch of f.

Two germs $f, g : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ are $\mathcal{R}$-equivalent if there exists a germ of diffeomorphism $\varphi : (\mathbb{K}^n, S) \to (\mathbb{K}^n, S)$ such that $g = f \circ \varphi^{-1}$. They are $\mathcal{L}$-equivalent if there exists a germ of diffeomorphism $\psi : (\mathbb{K}^p, 0) \to (\mathbb{K}^p, 0)$ such that $g = \psi \circ f$. We denote by $\mathrm{Diff}(\mathbb{K}^n, S)$ the group of germs of diffeomorphisms $\varphi : (\mathbb{K}^n, S) \to (\mathbb{K}^n, S)$ and $\mathrm{Diff}(\mathbb{K}^p, 0)$ the group of germs of diffeomorphisms $\psi : (\mathbb{K}^p, 0) \to (\mathbb{K}^p, 0)$.

Definition 2.0.1. *Two germs $f, g : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ are $\mathcal{A}$-equivalent ($f \sim_{\mathcal{A}} g$) if there exist diffeomorphisms $\varphi : (\mathbb{K}^n, S) \to (\mathbb{K}^n, S)$ and $\psi : (\mathbb{K}^p, 0) \to (\mathbb{K}^p, 0)$ such that $g = \psi \circ f \circ \varphi^{-1}$, that is, following diagram commutes*

$$\begin{tikzcd}
(\mathbb{K}^n, S) \arrow[r, "f"] \arrow[d, "\varphi"'] & (\mathbb{K}^p, 0) \arrow[d, "\psi"] \\
(\mathbb{K}^n, S) \arrow[r, "g"'] & (\mathbb{K}^p, 0)
\end{tikzcd}$$

Denoting by $\mathcal{A} = \mathrm{Diff}(\mathbb{K}^n, S) \times \mathrm{Diff}(\mathbb{K}^p, 0)$ we get a group which acts on $\mathcal{M}_n\mathcal{O}_{n,S}^p$ by $(\varphi, \psi) \cdot f = \psi \circ f \circ \varphi^{-1}$. In fact, $\mathcal{A} = \mathcal{R} \times \mathcal{L}$.

The contact group $\mathcal{K}$ is the set of germs of diffeomorphisms of $(\mathbb{K}^n \times \mathbb{K}^p, S \times \{0\})$ which can be written in the form $H(x, y) = (h(x), H_1(x, y))$, with $h \in \mathrm{Diff}(\mathbb{K}^n, 0)$ and $H_1(x, 0) = 0$ for x near 0.

Definition 2.0.2. *Two germs $f, g : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ are $\mathcal{K}$-equivalent if there exists $H \in \mathcal{K}$ such that $H(x, f(x)) = (h(x), g(h(x)))$.*

From the definition, $H(graph(f)) = graph(g)$ and H preserves the 0-fibres. Clearly $\mathcal{K}$-equivalence implies $\mathcal{A}$-equivalence, it is much broader, and it is used when we want to preserve the type of contact of germs with certain varieties or between different branches.

Example 2.0.3. *The germs $f(x, y) = (x, y^3)$ and $g(x, y) = (x, y^3 + xy)$ are $\mathcal{K}$-equivalent but not $\mathcal{A}$-equivalent.*

Definition 2.0.4. *Let $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$. An s-parameter unfolding is a germ $F : (\mathbb{K}^n \times \mathbb{K}^s, S \times \{0\}) \to (\mathbb{K}^p \times \mathbb{K}^s, 0)$ such that $F(x, \lambda) = (f_\lambda(x), \lambda)$ with $f_0(x) = f(x)$ for all $x \in (\mathbb{K}^n, S)$.*

Definition 2.0.5. *Two s-parameter unfoldings F, G of f are isomorphic if there exist diffeomorphisms $\Phi : (\mathbb{K}^n \times \mathbb{K}^s, S \times \{0\}) \to (\mathbb{K}^n \times \mathbb{K}^s, S \times \{0\})$ and $\Psi(\mathbb{K}^p \times \mathbb{K}^s, 0) \to (\mathbb{K}^p \times \mathbb{K}^s, 0)$ which are unfoldings of the identity in $(\mathbb{K}^n, S)$ and $(\mathbb{K}^p, 0)$ respectively such that $G = \Psi \circ F \circ \Phi^{-1}$.*

An unfolding F is trivial if it is isomorphic to the product unfolding $f \times id$.

We have $\Phi(x, \lambda) = (\phi_\lambda(x), x)$ with $\phi_0(x) = id(x)$ and $\Psi(y, \lambda) = (\psi_\lambda(y), y)$ with $\psi_0(y) = id(y)$. If $G(x, \lambda) = (g_\lambda(x), x)$, from the definition it follows that $g_\lambda = \psi_\lambda \circ f_\lambda \circ \phi_\lambda^{-1}$ as maps, not necessarily as germs.

Definition 2.0.6. *Let F be a s-parameter unfolding of f and $h : (\mathbb{K}^l, 0) \to (\mathbb{K}^s, 0))$, $h(v) = u$. Then*

$$\begin{aligned} G = h^*F : (\mathbb{K}^n \times \mathbb{R}^l, 0) &\longrightarrow (\mathbb{K}^p \times \mathbb{R}^l, 0) \\ (x, v) &\longmapsto (f(x, h(v)), v) \end{aligned}$$

is an l-parameter unfolding of f called the pullback of F by h.

Definition 2.0.7. *Two s-parameter unfoldings F, G of f are equivalent if there exists a germ of diffeomorphism $h : (\mathbb{K}^s, 0) \to (\mathbb{K}^s, 0))$ such that G is isomorphic to h^*F.*

Definition 2.0.8. *An unfolding F of f is* ***versal*** *if every unfolding of f is isomorphic to h^*F for some map h. A miniversal unfolding is a versal unfolding with a minimum number of parameters.*

Definition 2.0.9. *A germ $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ is stable if any unfolding F of it is trivial.*

This implies that f_λ is $\mathcal{A}$-equivalent to f (as a map, ϕ_λ and ψ_λ do not preserve S and the origin, respectively).

Let $\theta_{\mathbb{K}^n,S}$ and $\theta_{\mathbb{K}^p,0}$ be the $\mathcal{O}_n$-module of germs at S of vector fields on $\mathbb{K}^n$ and $\mathcal{O}_p$-module of germs at 0 of vector fields on $\mathbb{K}^p$ respectively. We denote them by θ_n and θ_p. Let $\theta(f)$ be the $\mathcal{O}_n$-module of germs $\xi : (\mathbb{K}^n, S) \to T\mathbb{K}^p$ such that $\pi_p \circ \xi = f$ where $\pi_p : T\mathbb{K}^p \to \mathbb{K}^p$ denotes the tangent bundle over $\mathbb{K}^p$. Therefore, $\theta(f) \cong \mathcal{O}_n^p \oplus \ldots \oplus \mathcal{O}_n^p$ (r-times). Now, given $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$, we define $f^* : \mathcal{O}_p \to \mathcal{O}_n$ such that $f^*(h) = h \circ f$, then $\theta(f)$ is an $\mathcal{O}_p$-module via f^*.

Define $tf : \theta_n \to \theta(f)$ by $tf(\xi) = df \circ \xi$ and $wf : \theta_p \to \theta(f)$ by $wf(\eta) = \eta \circ f$. The $\mathcal{A}_e$-tangent space of a germ f is defined as $T\mathcal{A}_e f = tf(\theta_n) + wf(\theta_p)$. We have that $tf(\theta_n)$ is an $\mathcal{O}_n$-submodule of $\theta(f)$, but it also has an $\mathcal{O}_p$-module structure via f^* and $wf(\theta_p)$ is an $\mathcal{O}_p$-submodule of $\theta(f)$ via f^*. Therefore, $T\mathcal{A}_e f$ is an $\mathcal{O}_p$-module via f^*.

The $\mathcal{K}_e$-tangent space of a germ f is defined as

$$T\mathcal{K}_e\, f = tf(\theta_n) + f^*(\mathcal{M}_p)\theta(f).$$

Definition 2.0.10. *The $\mathcal{A}_e$-codimension of a germ f, denoted by $\mathcal{A}_e$-cod(f), is the $\mathbb{K}$-vector space dimension of*

$$N\mathcal{A}_e(f) = \frac{\theta(f)}{T\mathcal{A}_e f}.$$

If we consider the $\mathcal{A}$-tangent space $T\mathcal{A}f = tf(\mathcal{M}_n \cdot \theta_n) + wf(\mathcal{M}_p \cdot \theta_p)$, we similarly define the $\mathcal{A}$-codimension of f as $\dim_{\mathbb{K}} \frac{\mathcal{M}_n\theta(f)}{T\mathcal{A}f}$.

We define $\mathcal{K}_e$-cod(f) similarly.

Theorem 2.0.11. *(Mather's infintesimal criterion, [14]) A germ $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ is stable if and only if $\mathcal{A}_e$-cod$(f) = 0$.*

This means that if f is stable, any vector field in $\theta(f)$ belongs to the tangent space of f.

Example 2.0.12. *i) $f(x, y) = (x, y^2)$. We will calculate $T\mathcal{A}_e f$. Here $\theta(f) \cong \mathcal{O}_2^2$. Let $\xi = (\xi_1, \xi_2) \in \theta_2$ be a vector field in the source and $\eta = (\eta_1, \eta_2) \in \theta_2$ a vector field in the target. Writing vector fields in columns, the tangent space is comprised by vector fields of type*

$$\begin{pmatrix} 1 & 0 \\ 0 & 2y \end{pmatrix} \begin{pmatrix} \xi_1(x,y) \\ \xi_2(x,y) \end{pmatrix} + \begin{pmatrix} \eta_1(x,y^2) \\ \eta_2(x,y^2) \end{pmatrix} = \begin{pmatrix} \xi_1(x,y) \\ 2y\xi_2(x,y) \end{pmatrix} + \begin{pmatrix} \eta_1(x,y^2) \\ \eta_2(x,y^2) \end{pmatrix}$$

From η_1 and η_2 we have constant terms in first and second rows. From ξ_1 we have any function in x and y in first row. From $2y\xi_2$ we have any function *with y in second row, so we are only missing pure terms in x in second row, which we can obtain from η_2. Therefore, the codimension of f is 0, i.e. it is a stable germ (which we already knew from Whitney [36]).*

ii) The bigerm $f(x) = \{f_1, f_2\} = \{(0,x),(x^2,x^3)\}$. Here $\theta(f) \cong \mathcal{O}^2_{1,S} \cong \mathcal{O}^2_1 \oplus \mathcal{O}^2_1$. Let $\xi^1, \xi^2 \in \mathcal{O}_1$ and $\eta = (\eta_1, \eta_2) \in \mathcal{O}_2$. Writing vector fields in columns, the tangent space is comprised by vector fields of type

$$\begin{pmatrix} df_1(\xi^1) & df_2(\xi^2) \end{pmatrix} + \begin{pmatrix} \eta_1(0,x) & \eta_1(x^2,x^3) \\ \eta_2(0,x) & \eta_2(x^2,x^3) \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 2x\xi^2(x) \\ \xi^1(x) & 3x^2\xi^2(x) \end{pmatrix} + \begin{pmatrix} \eta_1(0,x) & \eta_1(x^2,x^3) \\ \eta_2(0,x) & \eta_2(x^2,x^3) \end{pmatrix}$$

From ξ^1, η_1 and η_2 we have comprised by vector fields of type

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

so we are missing one constant vector field. From ξ^1 we get any term in x in position (2,1). Fixing $\eta_i(X,Y) = X^mY^n$ with $m > 0$ we get x^k, $k \neq 1,3$ in positions (1,2) and (2,2). Fixing $\xi^2(x) = x^2$, since we have x^4 in position (2,2), we get x^3 in position (1,2), and with $\xi^2 = 1$, since we have x^2 in position (2,2), we get x in position (1,2). Similarly we obtain x^3 in position (2,2). Having everything in position (1,2), by fixing $\eta_1(X,Y) = Y^r$ we get everything in position (1,1). Therefore, we are missing, besides one constant vector field, x in position (2,2), and so the codimension is 2.

iii) $f(x,y) = (x, y^3 + xy)$. Exercise (things are not as easy as it seems).

In general it is very difficult to calculate the codimension of a germ due to the module structure of the tangent space. We have to use the fact that the germ is finitely determined and ignore terms of order greater than its degree of determinacy. As we will see, formulas and alternative methods can be extremely helpful for this task.

Theorem 2.0.13. *(Mather) Let $F : (\mathbb{K}^n \times \mathbb{K}^s, S \times \{0\}) \to (\mathbb{K}^p \times \mathbb{K}^s, 0)$ be an s-parameter unfolding of a germ f, such that $F(x,\lambda) = (f_\lambda(x), \lambda)$ with $f_0(x) = f(x)$. Then F is versal if and only if*

$$T\mathcal{A}_e f + Sp_{\mathbb{K}}\{\frac{\partial f_\lambda}{\partial \lambda_1}, \dots, \frac{\partial f_\lambda}{\partial \lambda_r}\} = \theta(f).$$

If furthermore, $r = \mathcal{A}_e$-cod(f), then it is miniversal.

A map germ f is said to be k-determined if any germ g such that $j^k g = j^k f$, where j^k denotes the k-th order Taylor expansion, is $\mathcal{A}$-equivalent to f. If f is k-determined for some $k < \infty$, then we say it is finitely determined.

Theorem 2.0.14. *(Mather, [15]) A germ $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ is finitely determined if and only if $\mathcal{A}_e$-cod$(f) < \infty$.*

Given $f = \{f_1, \dots, f_r\} : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$, let $m_0(f) = \dim_{\mathbb{K}} \frac{\mathcal{O}_{n,S}}{f^*(\mathcal{M}_p)}$ denote the multiplicity of the germ f. Note that

$$\dim_{\mathbb{K}} \frac{\mathcal{O}_{n,S}}{f^*(\mathcal{M}_p)} = \sum_{i=1}^{r} \dim_{\mathbb{K}} \frac{\mathcal{O}_{n,x_i}}{f_i^*(\mathcal{M}_p)}.$$

A monogerm $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ is said to be of corank k if $df(0)$ has corank k. A multigerm $f = \{f_1, \dots, f_r\}$ is said to be of corank k if f_i is of corank less than or equal to k, $i = 1, \dots, r$.

A multigerm $f = \{f_1, \dots, f_r\} : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ with $S = \{x_1, \dots, x_r\}$ is simple if there exists a finite number of $\mathcal{A}$-classes (classes under the action of germs of diffeomorphisms in the source and target) such that for every unfolding $F : (\mathbb{K}^n \times \mathbb{K}^d, S \times \{0\}) \to (\mathbb{K}^p \times \mathbb{K}^d, 0)$ with $F(x, \lambda) = (f_\lambda(x), \lambda)$ and $f_0 = f$ there exists a sufficiently small neighbourhood U of $S \times \{0\}$ such that for every $(y_1, \lambda), \dots, (y_r, \lambda) \in U$ where $F(y_1, \lambda) = \dots = F(y_r, \lambda)$ the multigerm $f_\lambda : (\mathbb{K}^n, \{y_1, \dots, y_r\}) \to (\mathbb{K}^p, f_\lambda(y_i))$ lies in one of those finite classes.

Definition 2.0.15. *i) A vector field germ $\eta \in \theta_p$ is called liftable over f if there exists $\xi \in \theta_n$ such that $df \circ \xi = \eta \circ f$ ($tf(\xi) = wf(\eta)$). The set of vector field germs liftable over f is denoted by $Lift(f)$ and is an $\mathcal{O}_p$-module.*

ii) Let $\widetilde{\tau}(f) = ev_0(Lift(f))$ be the evaluation at the origin of elements of $Lift(f)$.

Consider $(V, 0)$ a germ of a variety in $(\mathbb{K}^n, 0)$ and $I \subset \mathcal{O}_n$ an ideal defining V. We define $Derlog(V) = \{\xi \in \theta(n) \,/\, \xi \cdot \tilde{h} \in I, \ \forall\, \tilde{h} \in I\}$.

In general $Lift(f) \subseteq Derlog(V)$ when V is the discriminant of an analytic f and $Derlog(V)$ represents the $\mathcal{O}_p$-module of vector fields tangent to V. We have an equality when $\mathbb{K} = \mathbb{C}$, f is complex analytic and f is finitely determined.

Example 2.0.16. *Consider the germ $f(x) = (x^2, 0) = (X, Y)$, which is not finitely determined (exercise). The image is given implicitly by the quasihomogeneous polynomial $Y = 0$, therefore, $Derlog(f) = \langle \frac{\partial}{\partial X}, Y\frac{\partial}{\partial Y} \rangle$. It is easy to see that the first generator is not liftable, in fact in [21] it is seen (and it can be checked) that $Lift(f) = \langle X\frac{\partial}{\partial X}, Y\frac{\partial}{\partial X}, Y\frac{\partial}{\partial Y} \rangle$.*

The set $\widetilde{\tau}(f)$ is the tangent space to the well defined manifold in the target containing 0 along which the map f is trivial (i.e. the analytic stratum). A finite set $E_1, \ldots, E_r$ of vector subspaces of a finite-dimensional vector space F has almost regular intersection of order k (with respect to F) if

$$\operatorname{cod}(E_1 \cap \ldots \cap E_r) = \operatorname{cod} E_1 + \ldots + \operatorname{cod} E_r - k,$$

where cod represents the codimension. When $k = 0$ we say regular intersection and when $k = 1$ we say almost regular intersection.

Theorem 2.0.17. *(Mather, [16]) The multigerm $f = \{f_1, \ldots, f_r\}$ is stable if and only if each f_i is stable and $\widetilde{\tau}(f_1), \ldots, \widetilde{\tau}(f_r)$ have regular intersection.*

Theorem 2.0.18. *(Cooper, Mond, Wik-Atique [6]) If $f = \{f_1, \ldots, f_r\}$ has $\mathcal{A}_e$-codimension 1 then each f_i is stable and $\widetilde{\tau}(f_1), \ldots, \widetilde{\tau}(f_r)$ have almost regular intersection:*

$$cod(\widetilde{\tau}(f_1) \cap \ldots \cap \widetilde{\tau}(f_r)) = \sum_{i=1}^{r} cod\,\widetilde{\tau}(f_i) - 1.$$

Example 2.0.19. *If f is a bigerm ($r = 2$) from $\mathbb{K}^2$ to $\mathbb{K}^3$ such that each branch is an immersion, then f is either $\mathcal{A}$-equivalent to:*

$$\begin{cases} (x, y) \mapsto (x, y, 0) \\ (x, y) \mapsto (x, 0, y) \end{cases}$$

which is stable since $\widetilde{\tau}(f_1) = \{(X, Y, Z) \,/\, Z = 0\}$ and $\widetilde{\tau}(f_2) = \{(X, Y, Z) \,/\, Y = 0\}$, or

$$\begin{cases} (x, y) \mapsto (x, y, 0) \\ (x, y) \mapsto (x, y, \varphi(x, y)) \end{cases}$$

where φ is called the separation function. D. Mond proved in [19] that bigerms of immersions are classified for $\mathcal{A}$ by the $\mathcal{K}$-class of the separation function. So $\varphi(x, y) = x^2 \pm y^{k+1}$ (A_k); $\varphi(x, y) = x^2 y \pm y^{k-1}$ (D_k); $\varphi(x, y) = x^3 + y^4$ (E_6); $\varphi(x, y) = x^3 \pm xy^3$ (E_7); $\varphi(x, y) = x^3 + y^5$ (E_8). Also, $\mathcal{A}_e$-cod(f)=$\mathcal{K}_e$-cod(φ). Note that $\widetilde{\tau}(f_1) = \widetilde{\tau}(f_2) = \{(X, Y, Z) \,/\, Z = 0\}$ in all these cases, therefore they have almost regular intersection.

Chapter 3

Multigerms of codimension 1

3.1 Reduction to $\mathcal{K}_V$-classification

Consider $(V,0)$ a germ of an analytic space in $(\mathbb{C}^p,0)$ and $I \subset \mathcal{O}_p$ an ideal defining V. Let $\mathcal{R}_V$ be the subgroup of $\mathcal{R}$ consisting of all germs $\varphi \in \mathcal{R}$ such that $\varphi^*(I) = I$, that is, $\varphi(V) = V$. We define $\mathcal{K}_V$ to be the subgroup of $\mathcal{K}$ consisting of germs $H \in \mathcal{K}$, $H(x,y) = (h(x), H_1(x,y))$, such that $h \in \mathcal{R}_V$. We set $\mathcal{R}_V^{(k)} = \mathcal{R}_V \cap \mathcal{R}^{(k)}$, where $\mathcal{R}^{(k)} = \{\varphi \in \mathcal{R} \,/\, \varphi = id \,(\bmod \mathcal{M}_p^{k+1})\}$. By results of R. Pellikaan [28], $J^k\mathcal{R}_V = \mathcal{R}_V \,/\, \mathcal{R}_V^{(k)}$, the set of k-jets of elements of $\mathcal{R}_V$, is a Lie group acting smoothly on $J^k(p,1)$. We define $J^k\mathcal{K}_V$ similarly. The results of this section can be found in [34].

The following result reduces the $\mathcal{A}$-classification of multigerms to the $\mathcal{K}_V$-classification of function germs. Let $T = \{x_1, \ldots, x_{r-1}\}$, then $S = T \cup \{x_r\}$.

Theorem 3.1.1. *Let $f = \{f_1, \ldots, f_{r-1}\} : (\mathbb{C}^n, T) \to (\mathbb{C}^{n+1}, 0)$ and $g, \tilde{g} : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be $\mathcal{A}$-finite multigerms such that $g = \{f, g_r\}$ and $\tilde{g} = \{f, \tilde{g}_r\}$. If $h, \tilde{h} \in \mathcal{O}_{n+1}$ are reduced defining equations for the images of g_r and $\tilde{g}_r$ respectively and V is the image of f, then g and $\tilde{g}$ are $\mathcal{A}$-equivalent if h and $\tilde{h}$ are $\mathcal{K}_V$-equivalent. The converse is true if the branches of g are inequivalent.*

Remark 3.1.2. *Notice that once we have the $\mathcal{K}_V$-orbits, the $\mathcal{A}$-classification follows as a quotient under the action of the permutation group on the points of S. In particular, modality in the $\mathcal{K}_V$-classification implies modality in the $\mathcal{A}$-classification.*

In order to obtain the $\mathcal{K}_V$-orbits we need to know the tangent spaces (see [4]):

Proposition 3.1.3. *Let $I \subseteq \mathcal{O}_p$ be an ideal defining V and $h \in \mathcal{O}_p$. Then $T\mathcal{K}_{V,e}\, h = th(Derlog(V)) + \langle h \rangle$ and $T\mathcal{K}_V\, h = th(\mathcal{M}_p\theta_p \cap Derlog(V)) + \langle h \rangle$, where $Derlog(V) = \{\xi \in \theta_p \,/\, \xi \cdot \tilde{h} \in I,\ \forall\, \tilde{h} \in I\}$.*

A criterion for finite determinacy can be obtained by imitating the $\mathcal{K}$-case (see [32]):

Theorem 3.1.4. *If $\mathcal{M}_p^{k+1} \subset \mathcal{M}_p\, T\,\mathcal{R}_V\, h + h^*\mathcal{M}_1 \cdot \mathcal{M}_p + \mathcal{M}_p^{k+2}$ then h is $k - \mathcal{K}_V$-determined.*

Given a germ f and a group $\mathcal{G}$, we set $N\mathcal{G}_e f = \theta(f)\,/\,T\,\mathcal{G}_e f$. The $\mathcal{G}_e$-codimension of f is the dimension of $N\mathcal{G}_e f$.

Theorem 3.1.5. *Let $f : (\mathbb{C}^n, T) \to (\mathbb{C}^{n+1}, 0)$ and $g = \{f, g_r\} : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be $\mathcal{A}$-finite multigerms such that g_r is an immersion. If $h \in \mathcal{O}_{n+1}$ is a reduced defining equation for the image of g_r and V is the image of f, then the following sequence is exact:*

$$0 \to N\,\mathcal{K}_{V,e}\,h \to N\,\mathcal{A}_e\,g \to N\,\mathcal{A}_e\,f \to 0$$

Corollary 3.1.6. *Under the hypotheses of Theorem 3.1.5, if f is stable then $N\mathcal{K}_{V,e}h$ is isomorphic to $N\mathcal{A}_e g$.*

Finally, to calculate the $\mathcal{A}$-codimension we have the following result due to L. Wilson [35].

Proposition 3.1.7. *Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ be an $\mathcal{A}$-finite multigerm, $r = |S|$. If f is stable the $\mathcal{A}_e$-codimension is equal to 0. If the $\mathcal{A}_e$-codimension is strictly greater than 0 then the following relation holds:*

$$\mathcal{A}_e - codimension = \mathcal{A} - codimension + r(p - n) - p.$$

The following result is a powerful tool for classification of singularities.

Proposition 3.1.8 (Mather's Lemma)**.** *Let the Lie group G act smoothly on the manifold M, and suppose that the submanifold S has the following properties:*

1. *for all $x \in S$, $T_xG \cdot x \supseteq T_xS$;*
2. *the dimension of $G \cdot x$ is independent of the choice of $x \in S$;*
3. *S is connected.*

Then S is contained in a single G orbit.

Example 3.1.9. *If $f : (\mathbb{C}^2, S) \to (\mathbb{C}^3, 0)$ is a simple bigerm in which one of the branches is a cross-cap, $f_1(x, y) = (x, y^2, xy)$, then it is $\mathcal{A}$-equivalent to one of*

$$f:\begin{cases}(x,y)\mapsto(x,y^2,xy)\\(x,y)\mapsto(x^k,x,y)\end{cases}\qquad (k\geq 1)\quad S_0A_0|_k^1$$

$$f:\begin{cases}(x,y)\mapsto(x,y^2,xy)\\(x,y)\mapsto(x,x^k,y)\end{cases}\qquad (k\geq 2)\quad S_0A_0|_k^2$$

$$f:\begin{cases}(x,y)\mapsto(x,y^2,xy)\\(x,y)\mapsto(x,y,y^{k-1})\end{cases}\qquad (k\geq 3)\quad S_0A_0|_k^3$$

Let V be the image of the cross-cap. Then

$$Derlog(V)=\langle(X,-2Y,0),(0,2Z,X^2),(X,0,Z),(Z,0,XY)\rangle$$

The $\mathcal{A}$-classes of such bigerms are given by the $\mathcal{K}_V$-classes in $\mathcal{O}_3$. We use Mather's Lemma to classify the orbits in jet spaces. First we consider the action of $J^1\mathcal{K}_V$ on $J^1(3,1)$. Let $\sigma(X,Y,Z)=aX+bY+cZ$ be an element in $J^1(3,1)$. It follows from Proposition 3.1.3 that

$$T_\sigma J^1\mathcal{K}_V\cdot\sigma=(aX,aZ,bY,bZ,cZ)\;\mathcal{O}_3\;(mod\,\mathcal{M}_3^2)$$

There are five orbits in $J^1(3,1)$: those of $X-Y$, X, Y, Z and 0. If $h(X,Y,Z)=X-Y$, it follows from Theorem 3.1.4 that h is $1-\mathcal{K}_V$- determined. Also h has $\mathcal{K}_{V,e}$-codimension 1.

We consider now germs whose 1-jet is either X, Y or Z. Let

$$\begin{aligned}\sigma_1(X,Y,Z)&=X+\textstyle\sum_{i+j+\ell=k}\alpha_{i,j,\ell}\,X^iY^jZ^\ell\\ \sigma_2(X,Y,Z)&=Y+\textstyle\sum_{i+j+\ell=k}\beta_{i,j,\ell}\,X^iY^jZ^\ell\\ \sigma_3(X,Y,Z)&=Z+\textstyle\sum_{i+j+\ell=k}\gamma_{i,j,\ell}\,X^iY^jZ^\ell\end{aligned}$$

be elements in $J^k(3,1)$. Then

$$\begin{aligned}T_{\sigma_1}J^k\mathcal{K}_V\cdot\sigma_1&=(X,\alpha_{0,k,0}Y^k,Z)\;\mathcal{O}_3\;(mod\,\mathcal{M}_3^{k+1})\\ T_{\sigma_2}J^k\mathcal{K}_V\cdot\sigma_2&=(\beta_{k,0,0}X^k,Y,Z)\;\mathcal{O}_3\;(mod\,\mathcal{M}_3^{k+1})\\ T_{\sigma_3}J^k\mathcal{K}_V\cdot\sigma_3&=(X^2,XY,\gamma_{0,k,0}Y^k,Z)\;\mathcal{O}_3\;(mod\,\mathcal{M}_3^{k+1})\end{aligned}$$

We have the following $J^k\mathcal{K}_V$-orbits in $J^k(3,1)$: those of $X-Y^k$, $Y-X^k$ and $Z-Y^k$, $k\geq 2$. It follows from Theorem 3.1.4 that $h_1(X,Y,Z)=X-Y^k$, $h_2(X,Y,Z)=Y-X^k$ and $h_3(X,Y,Z)=Z-Y^k$ are $k-\mathcal{K}_V$-determined. Also h_1 and h_2 have $\mathcal{K}_{V,e}$-codimension k and h_3 has $\mathcal{K}_{V,e}$-codimension $k+1$.

Let M be the space of quadratic forms in $\mathbb{C}^3$. The dimension of M is 6. If $\sigma\in M$ then

$$\dim\;T_\sigma J^2\mathcal{K}_V\,\sigma\leq 5$$

Therefore the dimension of all orbits of $J^2\mathcal{K}_V$ in M is less than the dimension of M. Hence the orbits form a continuous family in M. Consequently a germ in $\mathcal{O}_3$ whose 1-jet is 0 cannot be $\mathcal{K}_V$-simple.

3.2 Complete transversals

Let $\mathcal{G}$ be a subgroup of one of standard Mather groups $\mathcal{R}$, $\mathcal{L}$, $\mathcal{A}$, $\mathcal{K}$ and define $\mathcal{G}_s$ to be the subgroup of $\mathcal{G}$ whose elements have the s-jet equal to the identity. The results of this section can be found in [3].

Proposition 3.2.1 (Complete Transversal to jets). *Let $\mathcal{G}$ be a Mather's group and denote the tangent space to the $J^{k+1}\mathcal{G}_1$-orbit of $j^{k+1}f$ by $L(J^{k+1}\ \mathcal{G}_1).j^{k+1}f$. Then given $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ and $T \subset H^{k+1}(n,p)$ such that*

$$H^{k+1}(n,p) \subset L(J^{k+1}\mathcal{G}_1).j^{k+1}f + T$$

any $(k+1)$-jet $j^{k+1}g$, with $j^k g = j^k f$, is in the same $\mathcal{G}_1$-orbit of $j^{k+1}f + t$, for some $t \in T$.

Definition 3.2.2. *The set T is called a $(k+1)$-complete transversal.*

3.3 Classification of multigerms of $\mathcal{A}_e$-codimension 1 and corank 1

3.3.1 Augmentations

Some results of this section can be found in [6].

Definition 3.3.1. *Let $h : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ be a map-germ with a 1-parameter unfolding $H : (\mathbb{K}^n \times \mathbb{K}, S \times \{0\}) \to (\mathbb{K}^p \times \mathbb{K}, 0)$ which is stable as a map-germ, where $H(x, \lambda) = (h_\lambda(x), \lambda)$, such that $h_0 = h$. Let $g : (\mathbb{K}^q, 0) \to (\mathbb{K}, 0)$ be a function-germ. Then, the augmentation of h by H and g is the germ $A_{H,g}(h)$ given by $(x, z) \mapsto (h_{g(z)}(x), z)$. A multigerm that is not an augmentation is called* **primitive***.*

Theorem 3.3.2. *([10],[11])*

$$\mathcal{A}_e - cod(A_{H,g}(h)) \geq \mathcal{A}_e - cod(h)\tau(g),$$

where τ is the Tjurina number and equality is reached when g is quasihomogeneous.

Theorem 3.3.3. *([11]) Suppose that $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ is non-stable and has a 1-parameter stable unfolding F. Then*

$$q = \dim_{\mathbb{K}} \widetilde{\tau}(F) \geq 1 \Leftrightarrow f \text{ is an augmentation.}$$

More precisely, on the right hand side, $f \sim_{\mathcal{A}} A_{H,g}(h)$ for some $h : (\mathbb{K}^{n-q}, S') \to (\mathbb{K}^{p-q}, 0)$, a smooth map-germ with a 1-parameter stable unfolding H, and $g : (\mathbb{K}^q, 0) \to (\mathbb{K}, 0)$ a function, $q \geq 1$.

Proposition 3.3.4. *When $\mathcal{A}_e - cod(f) = 1$ and $g(z) = z^2$, the $\mathcal{A}$-equivalence class of $A_{F,g}(f)$ is independent of the choice of miniversal unfolding F of f.*

Example 3.3.5. The five $\mathcal{A}_e$-codimension 1 multigerms from $(\mathbb{C}^2, S)$ to $\mathbb{C}^3$ are:

I. S_1 (the birth of two umbrellas).

II. The non-transverse contact of two immersed sheets.

III. The intersection of three immersed sheets which are pairwise transverse, but with each one having first order tangency to the intersection of the other two.

IV. A cross-cap meeting an immersed plane.

V. A quadruple intersection.

IV and V are primitive. I is the augmentation of the cusp $t \mapsto (t^2, t^3)$, II is the augmentation of a tacnode (two curves simply tangent at a point), which itself is the augmentation of the map from two copies of $\mathbb{C}^0$ to $\mathbb{C}$ sending both points to $0 \in \mathbb{C}$, and III is the augmentation of three lines meeting pairwise transversely at a point.

The following result is a partial converse.

Proposition 3.3.6. *Suppose that $G = id_{\mathbb{C}} \times g_\lambda$ is a one dimensional stable unfolding of a multigerm $g = g_0$ and suppose that $h = id_{\mathbb{C}} \times g_{\lambda^2}$ has $\mathcal{A}_e$-codimension 1. Then g has $\mathcal{A}_e$-codimension 1 and G is a versal unfolding of g. Thus h is the augmentation of g.*

Given a stable map $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ let Pf (the 'prism' on f) be the trivial 1-parameter unfolding of f. We shall say that a map-germ is a **prism** if it is $\mathcal{A}$-equivalent to Pg for some germ g.

Lemma 3.3.7. *$T\mathcal{A}_e Pf = \mathcal{O}_{\mathbb{C}\times\mathbb{C}^n} \frac{\partial}{\partial \Lambda} \oplus \mathcal{O}_{\mathbb{C}} T\mathcal{A}_e f$.*

The multigerm f can be reconstructed, up to $\mathcal{A}$-equivalence, from Pf as the top arrow in the pullback of this diagram

$$\begin{array}{ccc} & & \mathbb{C}^p, 0 \\ & & \downarrow i \\ \mathbb{C} \times \mathbb{C}^n, \{0\} \times S & \xrightarrow{Pf} & \mathbb{C} \times \mathbb{C}^p, (0,0) \end{array}$$

Proposition 3.3.8. *Let $F(\lambda, x) = (\lambda, f_\lambda(x))$ be an $\mathcal{A}_e$-versal unfolding of an $\mathcal{A}_e$-codimension 1 multigerm f. Then $G(\mu, \lambda, x) = (\mu, \lambda, f_{\lambda^2+\mu}(x))$ is an $\mathcal{A}_e$-versal unfolding of $g = A_F f$.*

Since $G(\mu, \lambda, x) = (\mu, \lambda, f_{\lambda^2+\mu}(x))$ is an unfolding of $F(\mu, x) = (\mu, f_\mu(x))$ and F is stable then G is $\mathcal{A}$-equivalent to PF. Therefore if a multigerm is an augmentation, its miniversal unfolding is a prism. The converse is also true:

Theorem 3.3.9. *Let g be a multigerm of $\mathcal{A}_e$-codimension 1 and suppose that the miniversal unfolding G of g is a prism. Then g is an augmentation.*

3.3.2 Monic and binary concatenations

In this section we describe two basic operations, by which we "concatenate" stable unfoldings of germs to create new multigerms. There is no reason to require purity of dimension in a multigerm, and we allow different branches to have domains of different dimension. The results of this section can be found in [6].

The first concatenation operation is monic: from a multigerm with r branches we get a multigerm with $r+1$ branches, in which the extra branch is a fold or an immersion.

Theorem 3.3.10. *Let $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ be a multigerm of finite $\mathcal{A}_e$-codimension with a stable unfolding F on the single parameter t, let $0 \leq k \in \mathbb{Z}$ and let $g : (\mathbb{K}^p \times \mathbb{K}^k, 0) \to (\mathbb{K}^p \times \mathbb{K}, 0)$ be the fold map $(y, v) \mapsto (y, \sum_{j=1}^{k} v_j^2)$. Then*

1. *$\mathcal{A}_e - cod(g^*(F)) = \mathcal{A}_e\text{-}cod(f) = \mathcal{A}_e\text{-}cod(\{F, g\})$*
2. *both $g^*(F)$ and $\{F, g\}$ have 1-parameter stable unfoldings.*

Theorem 3.3.11. *Suppose that the germ f of Theorem 3.3.10 has $\mathcal{A}_e$- codimension 1. Then up to $\mathcal{A}$-equivalence, the bigerm $h = \{F, g\}$ obtained is independent of the choice of stable unfolding F.*

We denote the multigerm $\{F, g\}$ by $C_k(f)$.

Example 3.3.12. *The bigerm consisting of a cross cap together with an immersed plane transverse to the parametrisation of the cross-cap, and making contact of degree k with the double line in the cross-cap (see section [**?**] is obtained by applying C_0 to the germ $t \mapsto (t^2, t^{2k+1})$ parametrising the k-th order cusp.*

The second type of concatenation is a binary operation: given germs $f_0 : (\mathbb{K}^m, S) \to (\mathbb{K}^a, 0)$ and $g_0 : (\mathbb{K}^n, T) \to (\mathbb{K}^b, 0)$ with 1-parameter stable unfoldings F and G, we form the multigerm h essentially by putting together $\mathrm{id}_{\mathbb{K}^a} \times F$ and $G \times \mathrm{id}_{\mathbb{K}^b}$ so that their analytic strata meet subtransversely in $\mathbb{K}^{a+b+1}$.

Theorem 3.3.13. *Suppose the two map-germs $F(y, s) = (f_s(y), s)$ and $G(x, s) = (g_s(x), s)$ are stable, and let h be defined by*

$$\begin{cases} (X, y, u) \mapsto (X, f_u(y), u) \\ (x, Y, u) \mapsto (g_u(x), Y, u) \end{cases}.$$

Then provided $\mathcal{A}_e$-cod $(h) < \infty$, we have

1. *$\mathcal{A}_e$-cod $(h) \geq \mathcal{A}_e$-cod $(f_0) \times \mathcal{A}_e$-cod (g_0), with equality if and only if either $s \in ds(DerlogD(G))$ or $t \in dt(DerlogD(F))$, where $D(G)$ is the discriminant of G;*

2. *h has a 1-parameter stable unfolding;*

Now suppose both f_0 and g_0 have $\mathcal{A}_e$-codimension 1. By analogy with augmentation and the first type of concatenation, one would expect the result of this second type of concatenation to be independent, up to $\mathcal{A}$-equivalence, of the choice of stable unfoldings F and G. This is true over $\mathbb{C}$ but false over $\mathbb{R}$.

Example 3.3.14. *Let $f_0(y) = y^3,\ g_0(x) = x^3$, and take $F'(y,u) = (y^3+uy, u),\ F''(y,u) = (y^3 - yu, u), G(x,u) = (x^3+ux, u)$. Then the multi-germs*

$$h' : \begin{cases} (X,y,u) \mapsto (X, y^3+uy, u) \\ (x,Y,u) \mapsto (x^3+ux, Y, u) \end{cases}$$

and

$$h'' : \begin{cases} (X,y,u) \mapsto (X, y^3-uy, u) \\ (x,Y,u) \mapsto (x^3+ux, Y, u) \end{cases}$$

are not equivalent over $\mathbb{R}$.

Proposition 3.3.15. *Suppose that the germs f_0 and g_0 in Theorem 3.3.13 both have $\mathcal{A}_e$-codimension 1. Then over $\mathbb{C}$, and up to $\mathcal{A}$-equivalence, the germ h is independent of choice of the 1-parameter stable unfoldings F and G.*

Theorem 3.3.16. *(Cooper, Mond, Wik-Atique [6]) Let $h = \{f, g\}$ be a primitive $\mathcal{A}_e$-codimension 1 map-germ in the nice dimensions (with no submersive branches). Then f and g are both stable. Also*

1. *If f and g are not transverse, then h is equivalent to*

$$\begin{cases} (x_1,\ldots,x_n) \mapsto \sum_i x_i^2 \\ (y_1,\ldots,y_m) \mapsto \sum y_j^2 \end{cases}$$

   Now assume f is transverse to g.

2. *If g is not transverse to $\widetilde{\tau}(f)$, then f is transverse to $\widetilde{\tau}(g)$, and h is equivalent to*

$$\begin{cases} (x_1,\ldots,x_n,u) \mapsto (f_u(x), u) \\ (\lambda_1,\ldots,\lambda_{p-1}, v_1,\ldots,v_k) \mapsto (\lambda, \sum_i v_i^2) \end{cases}$$

   (so $\{f,g\}$ is equivalent to $C_k(f_0)$).

3. *If g is transverse to $\widetilde{\tau}(f)$ and f is transverse to $\widetilde{\tau}(g)$, then $\{f,g\}$ is equivalent to a germ of the form*

$$\begin{cases} (X,y,u) \mapsto (X, f_u(y), u) \\ (x,Y,v) \mapsto (g_{Y,v}(x), Y, v) \end{cases}$$

where the target is decomposed as $\mathbb{C}^a \times \mathbb{C}^b \times \mathbb{C}$*, and* f_0 *and* g_0 *are primitive. If also the pullback of* g *by* $\widetilde{\tau}(f)$ *or the pullback of* f *by* $\widetilde{\tau}(g)$ *is quasihomogeneous then* $\{f, g\}$ *is equivalent to*

$$\begin{cases} (X, y, u) \mapsto (X, f_u(y), u) \\ (x, Y, v) \mapsto (g_v(x), Y, v), \end{cases}$$

i.e. to $B(f_0, g_0)$.

Chapter 4

Multigerms of codimension 2

In this chapter two new operations are introduced. Namely the simultaneous augmentation and concatenation and a generalised concatenation which includes the monic and binary concatenations as particular cases. We then show that any codimension 2 multigerm can be obtained using these operations and augmentations (with a few exceptions which are detailed in the main theorem). The results in this chapter can be found in [26] unless otherwise stated.

4.1 Looking for candidates

The following method provides candidates of corank 1 simple multigerms (without normal forms) for a certain codimension in a fixed pair (n, p) of source and target dimensions.

We are considering corank 1 multigerms of type $A_{k_1,\ldots,k_r}$, for which it is known that their corresponding orbits in the multijet space are defined by submersions in the stable case and by complete intersections in the finitely determined case ([7],[18]).

We repeat her the close relation between the $\mathcal{A}$-codimension and the $\mathcal{A}_e$-codimension due to Wilson's formula

$$\mathcal{A}_e - \operatorname{cod}(f) = \mathcal{A} - \operatorname{cod}(f) + r(p-n) - p,$$

where r is the number of branches.

Let $f = \{f_1, \ldots, f_r\} : (\mathbb{K}^n, S) \longrightarrow (\mathbb{K}^p, y)$ be a non-stable multigerm with $\mathcal{A}$-codimension s. Let's assume that f is k-determined and $\mathcal{A}$-simple. Suppose there exists a smooth submanifold $X \subset {}_rJ^k(\mathbb{K}^n, \mathbb{K}^p)$ such that for all $g : \mathbb{K}^n \longrightarrow \mathbb{K}^p$ and for all $\{z_1, \ldots, z_r\} \subset \mathbb{K}^n$ we have that ${}_rj^k g(z_1, \ldots, z_r) \in X$ if and only if the multigerm of g in $\{z_1, \ldots, z_r\}$ is $\mathcal{A}$-equivalent to f. We have:

Lemma 4.1.1. $\operatorname{cod}_{{}_rJ^k(\mathbb{K}^n,\mathbb{K}^p)} X = s + (r-1)p.$

Proof. This is proved by standard multijet and transversality techniques, for a detailed account see [24]. □

If the $\mathcal{A}$-codimension of f_j is i_j, $j = 1, \ldots, r$, this means that each f_j defines a smooth submanifold in the appropriate jet space of respective codimension i_j. These submanifolds are defined by $i_1, \ldots, i_r$ equations respectively.

If we consider the submanifold $X \subset {}_rJ^k(\mathbb{K}^n, \mathbb{K}^p)$ defined by the equations which define the multigerm (i.e. the equations which define each of the branches, which are independent since they involve different variables, plus the equations arising from all the points having the same image in the target space), we have that its codimension is $i_1 + \ldots + i_r + (r-1)p$ (the $(r-1)p$ extra equations come from $f(x_1) = ... = f(x_r)$). From the previous Lemma the codimension of such a submanifold is $s + (r-1)p$, so we deduce that the $\mathcal{A}$-codimension of the multigerm is $s = i_1 + \ldots + i_r$. In the case of some type of contact between the strata of the discriminant of different branches, other equations describing these contacts should be added to define the corresponding submanifold in the multijet space and so, in that case $s \geq i_1 + \ldots + i_r$.

Example 4.1.2. *i) Consider maps from $\mathbb{K}^2$ to $\mathbb{K}^2$. The only stable monogerms are the fold of $\mathcal{A}$-codimension 1 and the cusp of $\mathcal{A}$-codimension 2. To find all trigerms of $\mathcal{A}_e$-codimension 2 (which means $\mathcal{A}$-codimension 4) we set the equation $4 = i_1 + i_2 + i_3 + c$, where the i_j are the $\mathcal{A}$-codimension of the three branches and c corresponds to the number of equations defining contacts between the branches. The only possibilities are $i_1 = i_2 = i_3 = c = 1$ and $i_1 = i_2 = 1, i_3 = 2, c = 0$. These two respresent a trigerm with 3 fold branches, two of which have a first order tangency and two transversal fold branches together with a cusp transversal in the limit to both fold branches, respectively.*

ii) For $\mathcal{A}_e$-codimension 2 bigerms from $\mathbb{K}^3$ to $\mathbb{K}^3$ (which means $\mathcal{A}$-codimension 5) we set the equation $5 = i_1 + i_2 + c$. Now there are many more possibilities since $c \in \{0, 1, 2, 3\}$. For example, if $i_1 = 1, i_2 = 4, c = 0$ we have an $\mathcal{A}_e$-codimension 1 monogerm and a fold with transversality amongst all the strata. An other possibility is that $i_1 = i_2 = 2, c = 1$, which represents two cuspidal edges where the tangent line to one of the cuspidal edges is contained in the tangent plane in the limit to the other cuspidal edge (T^1_{22} in the notation of [25]).

In some cases, one same configuration may lead to different germs. If $i_1 = 1, i_2 = 2, c = 2$ we have a fold and a cuspidal edge, but $c = 2$ may represent a tangency between the tangent plane in the limit of the cuspidal edge and the fold surface (T^1_{12}) or a degenerate tangency between the cuspidal edge curve and the fold surface DT_{12}.

4.2 Simultaneous augmentation and concatenation

The operations introduced in the previous Section provide complete lists of $\mathcal{A}_e$-codimension 1 germs but the might fail to give complete lists for $\mathcal{A}_e$-codimension 2 as we will see in the next example for which we will need two previous results:

Lemma 4.2.1. *Let $f = \{f_1, \dots, f_r\} : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ be a stable corank 1 germ. Then $m_0(f) \leq n + r$ if $n = p$ and $m_0(f) \leq [\frac{n+1+r}{2}]$ if $p = n + 1$.*

From ii) in Example 4.1.2, as a candidate of an $\mathcal{A}_e$-codimension 2 bigerm from $\mathbb{K}^3$ to $\mathbb{K}^3$ we get an $\mathcal{A}_e$-codimension 1 monogerm and a fold. Lets take for example 4_1^2 from the list in [17], i.e. $f(x, y, z) = (x^4 + yx + z^2x^2, y, z)$ and consider the bigerm

$$h = \{f, g\} = \begin{cases} (x^4 + yx + z^2x^2, y, z) \\ (x, y, z^2) \end{cases}$$

which does, in fact, have codimension 2. We want to know wether this bigerm can be obtained by any of the previous operations. First of all, it cannot be a monic or binary concatenation since f is not stable. Next, to see wether h is an augmentation, we must check wether it admits a 1-parameter stable unfolding. To see wether a germ admits a 1-parameter stable unfolding or not we first see its multiplicity. An unfolding H of h will have the same multiplicity than h, which is 6=4+2, so h could admit a 1-parameter stable unfolding in principle. In fact

$$H = \{F, G\} = \begin{cases} (x^4 + yx + z^2x^2 + tx^2, y, z, t) \\ (x, y, z^2, t) \end{cases}$$

is a 1-parameter stable unfolding of h. However, F is a curve of swallowtail points and G is a fold 3-manifolds transversal to this curve, so $\widetilde{\tau}(F) = \{0\}$. Therefore, h is not an augmentation either.

In any case, it is clear that f is an augmentation of $f_0(x, y) = (x^4 + yx, y)$, and g is a fold similar to the one which appear in the monic concatenation. It is quite natural to define:

Definition 4.2.2. *Suppose $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ has a 1-parameter stable unfolding $F(x, \lambda) = (f_\lambda(x), \lambda)$. Let $g : (\mathbb{K}^p \times \mathbb{K}^{n-p+1}, 0) \to (\mathbb{K}^p \times \mathbb{K}, 0)$ be the fold map $(X, v) \mapsto (X, \Sigma_{j=p+1}^{n+1} v_j^2)$. Then, the multigerm $\{A_{F,\phi}(f), g\}$, where $\phi : \mathbb{K} \to \mathbb{K}$, is called the* simultaneous augmentation and monic concatenation *of the germ f by F and ϕ.*

This definition is consistent because of the following

Theorem 4.2.3. *Up to $\mathcal{A}$-equivalence, if $\mathbb{K} = \mathbb{C}$ and $\mathcal{A}_e - \mathrm{cod}(f) = 1$, the multigerm $\{Af, g\}$ is independent of the choice of miniversal unfolding F of f.*

It is important to be able to know the codimension of a multigerm produced by this operation.

Theorem 4.2.4.

$$\mathcal{A}_e - cod(\{A_{F,\phi}(f), g\}) \geq \mathcal{A}_e - cod(f)(\tau(\phi) + 1),$$

where τ is the Tjurina number of ϕ. Equality is reached when ϕ is quasi-homogeneous.

Example 4.2.5. *Let $f_l(x, y) = (x^3 + y^l x, y)$ and $F_l(x, y, z) = (x^3 + y^l x + zx, y, z)$ with augmentations $A^m F_l(x, y, z) = (x^3 + y^l x + z^m x, y, z)$ of codimension $(l-1)(m-1)$. The simultaneous augmentation and concatenation produces the bigerm*

$$\{A^m F_l, g\} = \begin{cases} (x^3 + y^l x + z^m x, y, z) \\ (x, y, z^2) \end{cases}$$

whose codimension is $(l - 1)(m - 1) + (l - 1) = (l - 1)m$.

Similarly to augmentations, simultaneous augmentations and concatenations admit 1-parameter stable unfoldings

Proposition 4.2.6. *The multigerm $\{A_{F,\phi'}(f), G\}$ where $\phi'(z, \mu) = \phi(z) + \mu$ and $G(X, v, \mu) = (X, \Sigma_{j=p+1}^{n+1} v_j^2, \mu)$ is a 1-parameter stable unfolding of $\{A_{F,\phi}(f), g\}$.*

Let $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ be a map germ and $F(x, \lambda) = (f_\lambda(x), \lambda)$ a 1-parameter unfolding. We say that F is a substantial unfolding if λ is contained in $d\lambda(Lift(F))$.

Theorem 4.2.7. *([5]) Suppose $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ satisfies the Mond conjecture and has a 1-parameter substantial stable unfolding $F(x, \lambda) = (f_\lambda(x), \lambda)$. Let $g : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1} \times \mathbb{C}, 0)$ be the immersion $X \mapsto (X, 0)$. Then, the multigerm $\{A_{F,\phi}(f), g\}$, where $\phi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, satisfies the Mond conjecture, i.e.*

$$\mathcal{A}_e - cod(\{A_{F,\phi}(f), g\}) \leq \mu_I(\{A_{F,\phi}(f), g\}).$$

Equality is reached if both f and ϕ are quasihomogeneous.

4.3 Generalised concatenation

One might ask at this point wether we have defined enough operations already to obtain all codimension 2 germs. The answer is no. In ii) of Example 4.1.2, another candidate for an $\mathcal{A}_e$-codimension 2 bigerm from $\mathbb{K}^3$ to $\mathbb{K}^3$ was T_{22}^1, two cuspidal edges where the tangent line to one of the cuspidal edge curves is contained in the tangent plane in the limit of the other cuspidal edge. Consider the $\mathcal{A}_e$-codimension 2 (exercise) bigerm

$$h = \{f, g\} = \begin{cases} (x^3 + y^2 x + zx, y, z) \\ (x, y, z^3 + yz) \end{cases} .$$

A 1-parameter stable unfolding H would have multiplicity 6 and would consist of two cuspidal edge surfaces, which in $\mathbb{R}^4$ intersect generically in one point. This means that the analytic stratum of H would be $\{0\}$, and therefore h is not an augmentation. It is not a monic concatenation or a simultaneous augmentation and concatenation because there is no fold.

Notice that from the definition of binary concatenation it follows that $f \pitchfork \widetilde{\tau}(g)$ and $g \pitchfork \widetilde{\tau}(f)$. The bigerm h satisfies that $f \pitchfork \widetilde{\tau}(g)$ but the tangent line to f is contained in the tangent plane in the limit of g, i.e. g is not transversal to $\widetilde{\tau}(f)$. This means that h is not a binary concatenation.

The idea is to generalize the concept of concatenation in order to be able to concatenate with singularities more degenerate than a fold or an immersion.

Definition 4.3.1. *Let $f : (\mathbb{K}^{n-s}, S) \to (\mathbb{K}^{p-s}, 0)$, $s < p$, be of finite $\mathcal{A}_e$-codimension and let $F : (\mathbb{K}^n, S \times \{0\}) \to (\mathbb{K}^p, 0)$ be a s-parameter stable unfolding of f with*

$$F(x_1, \dots, x_n) = (F_1(x_1, \dots, x_n), \dots, F_{p-s}(x_1, \dots, x_n), x_{n-s+1}, \dots, x_n),$$

where $F_i(x_1, \dots, x_{n-s}, 0, \dots, 0) = f_i(x_1, \dots, x_{n-s})$. Suppose that $\overline{g} : (\mathbb{K}^{n-p+s}, T) \to (\mathbb{K}^s, 0)$ is stable. Then the multigerm $\{F, g\}$ is a generalised concatenation of f with g, where $g = Id_{\mathbb{K}^{p-s}} \times \overline{g}$.

Observe that with this definition, $\dim \widetilde{\tau}(g) \geq p - s \geq 1$. If g is a monogerm and $\dim \widetilde{\tau}(g) = p - s$, it is of the form

$$g(x_1, \dots, x_n) = (x_1, \dots, x_{p-s}, g_{p-s+1}(x_{p-s+1}, \dots, x_n), \dots, g_p(x_{p-s+1}, \dots, x_n)).$$

Therefore, the definition implies that $F \pitchfork \widetilde{\tau}(g)$.

This definition is independent up to $\mathcal{A}$-equivalence of the choice of parametrisation of g as long as it is an $(p - s)$-parameter suspension of a germ $\overline{g}$. This was proved in [5] for a particular case of generalized concatenation but the proof can be adapted to the general case.

Proposition 4.3.2. *Given $\tilde{g} = id_{\mathbb{K}^{n-2}} \times \tilde{g}_0$, where $\tilde{g}_0$ is $\mathcal{A}$-equivalent to $\overline{g}$, there exists an s-parameter stable unfolding F' of f such that $\{F', \tilde{g}\}$ is $\mathcal{A}$-equivalent to $\{F, g\}$.*

Remark 4.3.3. *i) The monic concatenation is recovered by taking $s = 1$ and $g_p(x_p, \dots, x_n) = \Sigma_{i=p}^n x_i^2$ (or $g_p = 0$ when $n = p - 1$).*

ii) A binary concatenation $h = \{F, G\}$

$$\begin{cases} (X, y, u) \mapsto (f_u(y), u, X) \\ (x, Y, u) \mapsto (Y, u, g_u(x)) \end{cases}$$

is also a generalised concatenation where $p = a+1+b$, $n = b+m+1 = l+a+1$ and $s = b + 1$. In fact, the first branch is a $b + 1$-parameter stable unfolding of an f_0 and $\widetilde{\tau}((u, g_u(x))) = \{0\}$ when g_0 is not stable.

This operation is very general and cannot be studied in such a form, so we study particular cases where the stable germ g is given. In [26] and [5] three examples are studied, namely the cuspidal concatenation, the double fold concatenation and the cross-cap concatenation.

Theorem 4.3.4. *Consider $f : (\mathbb{K}^{n-2}, S) \to (\mathbb{K}^{n-2}, 0)$ with $n \geq 3$, $F(x, \lambda) = (f_\lambda(x), \lambda)$ a 2-parameter stable unfolding of f and*

$$g(x_1, \ldots, x_{n-2}, y, z) = (x_1, \ldots, x_{n-2}, y, z^3 + yz)$$

being a suspension of a cusp. We call the multigerm $\{F, g\}$ the cuspidal concatenation of f.

We have

$$\mathcal{A}_e - cod(\{F, g\}) = \dim_{\mathbb{K}} \frac{\mathcal{O}_{n-1}}{\{\xi : \xi = -z\eta_{n-1}(x, -3z^2, -2z^3) + \eta_n(x, -3z^2, -2z^3)\}},$$

where η_{n-1} and η_n are the last two components of vector fields in $Lift(F)$.

Example 4.3.5. *Consider $f(x) = x^3$ and a 2-parameter stable unfolding $F(x, \lambda_1, \lambda_2) = (x^3 + \lambda_1^2 x + \lambda_2 x, \lambda_1, \lambda_2)$. The cuspidal concatenation of f is the germ T_{22}^1 with which we began this section. Studying $Lift(F)$ we can prove that the $\mathcal{A}_e$-codimension is in fact 2.*

Another type of generalised concatenation is to concatenate with two fold hypersurfaces (in the equidimensional case, but the operation can be defined for $n \neq p$ too).

Theorem 4.3.6. *Let $f : (\mathbb{K}^{n-2}, S) \to (\mathbb{K}^{n-2}, 0)$ ($n \geq 3$) be a finitely determined germ, $F(x, \lambda) = (f_\lambda(x), \lambda)$ a 2-parameter stable unfolding of f and $g = \{g_1, g_2\}$ where*

$$\begin{cases} g_1(x_1, \ldots, x_{n-2}, y, z) = (x_1, \ldots, x_{n-2}, y, z^2) \\ g_2(x_1, \ldots, x_{n-2}, y, z) = (x_1, \ldots, x_{n-2}, y, z^2 + y). \end{cases}$$

We call the multigerm $\{F, g\}$ the double fold concatenation of f.

We have

$$\mathcal{A}_e - \mathrm{cod}(\{F, g\}) = \mathcal{A}_e - \mathrm{cod}(\{F, g_1\}) + \dim_{\mathbb{K}} \frac{\mathcal{O}_{n-1}}{\{\xi : \xi = -\eta_{n-1}(x, y, y) + \eta_n(x, y, y)\}},$$

where η_{n-1}, η_n are the last two components of vector fields in $Lift(\{F, g_1\})$.

For the case $n = p - 1$ we can define

Theorem 4.3.7. *([5]) Consider $f : (\mathbb{K}^{n-3}, S) \to (\mathbb{K}^{n-2})$ with $n \geq 3$, $F(x, \lambda) = (f_\lambda(x), \lambda)$ a 3-parameter stable unfolding of f and*

$$g(x_1, ..., x_{n-3}, y, z, w) = (x_1, ..., x_{n-3}, y, z, w^2, zw),$$

a suspension of a crosscap. We call the multigerm $\{F, g\}$ the crosscap concatenation of f.

We have

$$\mathcal{A}_e - cod(\{F, g\}) = dim_{\mathbb{K}} \frac{\mathcal{O}_n \oplus \mathcal{O}_n}{T_0},$$

where $T_0 = \{(\xi_1, \xi_2); \xi_1 = 2wv_n(x, y, z, w) + \eta_n(x, y, z, w^2, zw)$ *and* $\xi_2 = -w\eta_{n-1}(x, y, z, w^2, zw) + zv_n(x, y, z, w) + \eta_{n+1}(x, y, z, w^2, zw)\}$, η_{n-1}, η_n *and* η_{n+1} *are the last three components of vector fields in* $Lift(F)$ *and* $v_n \in \mathcal{O}_n$.

Example 4.3.8. *Let* $f(x) = (x^2, x^3)$ *and the family of* 3*-parameter stable unfoldings* $F_l(x, y, z, w) = (x^2, x^3 + xy^l + xz, y, z, w)$, $l \geq 1$. *Concatenating with a crosscap we obtain the bigerms*

$$\{F_l, g\} : \begin{cases} (x^2, x^3 + xy^l + xz, y, z, w) \\ (x, y, z, w^2, zw) \end{cases}.$$

Studying $Lift(F_l)$ *we can prove* $\mathcal{A}_e$*-cod*$(\{F_l, g\}) = l$.

Now consider the 3*-parameter stable unfolding:*

$$F_\infty : (x^2, x^3 + xy, y, z, w).$$

It can be seen that $\{F_\infty, g\}$ *is not finitely determined.*

4.4 $\mathcal{A}_e$-codimension 2 multigerms

We can now prove that we have defined enough operations to obtain all $\mathcal{A}_e$-codimension 2 multigerms. It is natural that in low dimensions, that is, $p \leq 2$, there will appear some special multigerms which cannot be obtained from any simpler germ. First we need the following

Proposition 4.4.1. *Let* $h = \{h_1, \ldots, h_r\} : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ $(p > 1)$ *be a multigerm of* $\mathcal{A}_e$*-codimension 2. Then*

i) If h_1 *is non stable, then* $r = 2$ *and* h_2 *is a prism on a Morse function or an immersion.*

ii) If $r \geq 3$ *then* h_i *is stable for every* $i \in \{1, \ldots, r\}$.

iii) If h_i *is stable for every* $i \in \{1, \ldots, r\}$, *then* $h = \{f, g\}$ *where both* f *and* g *are stable.*

This is not true for $p = 1$ since a trigerm of three Morse functions has codimension 2 and cannot be separated into two stable germs.

Finally we have the main result of this section.

Theorem 4.4.2. *Let $h = \{f, g\}$ be of $\mathcal{A}_e$-codimension 2, then*

1) *if f is a monogerm of $\mathcal{A}_e$-codimension 1, then g a prism on a Morse function or an immersion and*

   i) *h is an augmentation if and only if f is an augmentation with $m_0(f) \leq p$ when $n \geq p$ ($m_0(f) \leq [\frac{n}{2}]$ when $p = n+1$),*

   ii) *h is a simultaneous augmentation and concatenation if f is an augmentation with $m_0(f) = p+1$ when $n \geq p$ ($m_0(f) = [\frac{n}{2}]+1$ when $p = n+1$),*

   iii) *if $p = 1, 2$ and $m_0(f) = p+2$ when $n \geq p$ ($m_0(f) = [\frac{n}{2}] + 2$ when $p = n+1$) then f is a primitive monogerm of codimension 1,*

   iv) *if $(n, p) = (3, 4)$ and $m_0(f) = 3$ then h is $\mathcal{A}$-equivalent to*

$$\begin{cases} (u, v, x^3 + ux, x^4 + vx) \\ (u, v, u, x) \end{cases}$$

2) *if f and g are stable, then*

   i) $\mathrm{cod}(\widetilde{\tau}(f)) + \mathrm{cod}(\widetilde{\tau}(g)) \leq p$ *if and only if h is an augmentation,*

   ii) *if h is primitive and g is not transverse to $\widetilde{\tau}(f)$, then*

      a) *if $f \pitchfork g$, then*

         a1) *Suppose g is a monogerm. When $Im(dg_0) = \widetilde{\tau}(g)$, h is a monic concatenation. When $Im(dg_0) \supsetneqq \widetilde{\tau}(g)$, then either h is a (non-monic) generalised concatenation with g, it is a bigerm with two A_2-singularities or it is a trigerm of an A_2-singularity with two prisms on Morse functions (only if $n \geq p = 2$).*

         a2) *Suppose g is a multigerm, then it is a bigerm and either h is a double fold (immersion) concatenation with g or it is a trigerm of an A_2-singularity with two prisms on Morse functions (only if $n \geq p = 2$).*

      b) *If g and f are not transverse then f is a Morse function and g is an A_2-singularity (only if $n \geq p = 2$), h is a simultaneous augmentation and concatenation or, when $p = n+1$ and n even, it is $\mathcal{A}$-equivalent to*

$$n = 2 \begin{cases} (x, y^2, xy) \\ (x, x^2, y) \end{cases} \quad , \quad n = 4 \begin{cases} (u_1, v_1, v_2, y^3 + u_1 y, v_1 y + v_2 y^2) \\ (u_1, v_1, v_2, u_1^2 + v_2, y) \end{cases} \quad ,$$

$$n = 2k-2, k \geq 4 \begin{cases} (u_1, \ldots, u_{k-2}, v_1, \ldots, v_{k-1}, y^k + \sum_{i=1}^{k-2} u_i y^i, \sum_{i=1}^{k-1} v_i y^i) \\ (u_1, \ldots, u_{k-2}, v_1, \ldots, v_{k-2}, u_{k-3} + u_{k-2}^2, v_{k-1}, y) \end{cases} \quad .$$

iii) if h is primitive, $g \pitchfork \widetilde{\tau}(f)$ and $f \pitchfork \widetilde{\tau}(g)$, then h is a non-monic generalised concatenation.

We show an example of how to obtain all $\mathcal{A}_e$-codimension 2 multigerms from $\mathbb{K}^3$ to $\mathbb{K}^3$, which were first obtained in [25]. We start with $\mathcal{A}_e$-codimension 1 germs from $\mathbb{K}$ to $\mathbb{K}$ to obtain germs from $\mathbb{K}^2$ to $\mathbb{K}^2$, and then, these germs, together with the special germs in the Theorem and the $\mathcal{A}_e$-codimension 2 germs from $\mathbb{K}$ to $\mathbb{K}$ we obtain a complete list for $n = p = 3$. We know the list is complete because we have exhausted all the possibilities for the operations.

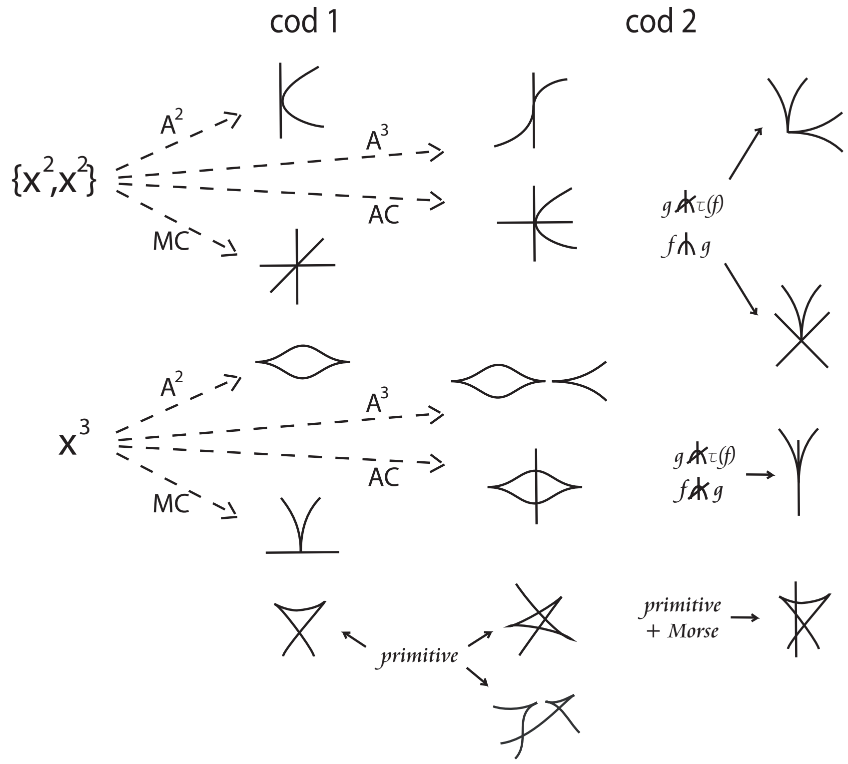

Figure 4.1: Codimension 1 and 2 germs and multigerms of maps from $\mathbb{C}^2$ to $\mathbb{C}^2$. The cases where a codimension 1 germ appears, a stabilisation is represented.

For example, from $\{x^2, x^2\}$, we take a 1-parameter stable unfolding given by $\{(x^2 + y, y), (x^2, y)\}$. We can augment it to obtain $\{(x^2 + y^k, y), (x^2, y)\}$ which is an ordinary tangency between to fold curves when $k = 2$ or a degenerate tangency when $k = 3$. We can also do a monic concatenation to obtain an ordinary triple point $\{(x^2 + y, y), (x^2, y), (x, y^2)\}$, and, finally, we can simultaneously augment and concatenate it to obtain the $\mathcal{A}_e$-codimension 2 trigerm $\{(x^2+y^2, y), (x^2, y), (x, y^2)\}$.

Notice that a same multigerm can be obtained from different germs by different operations. For example, take the $\mathcal{A}_e$-codimension 1 bigerm $\{(x^2, y), (x, y^3 + xy)\}$

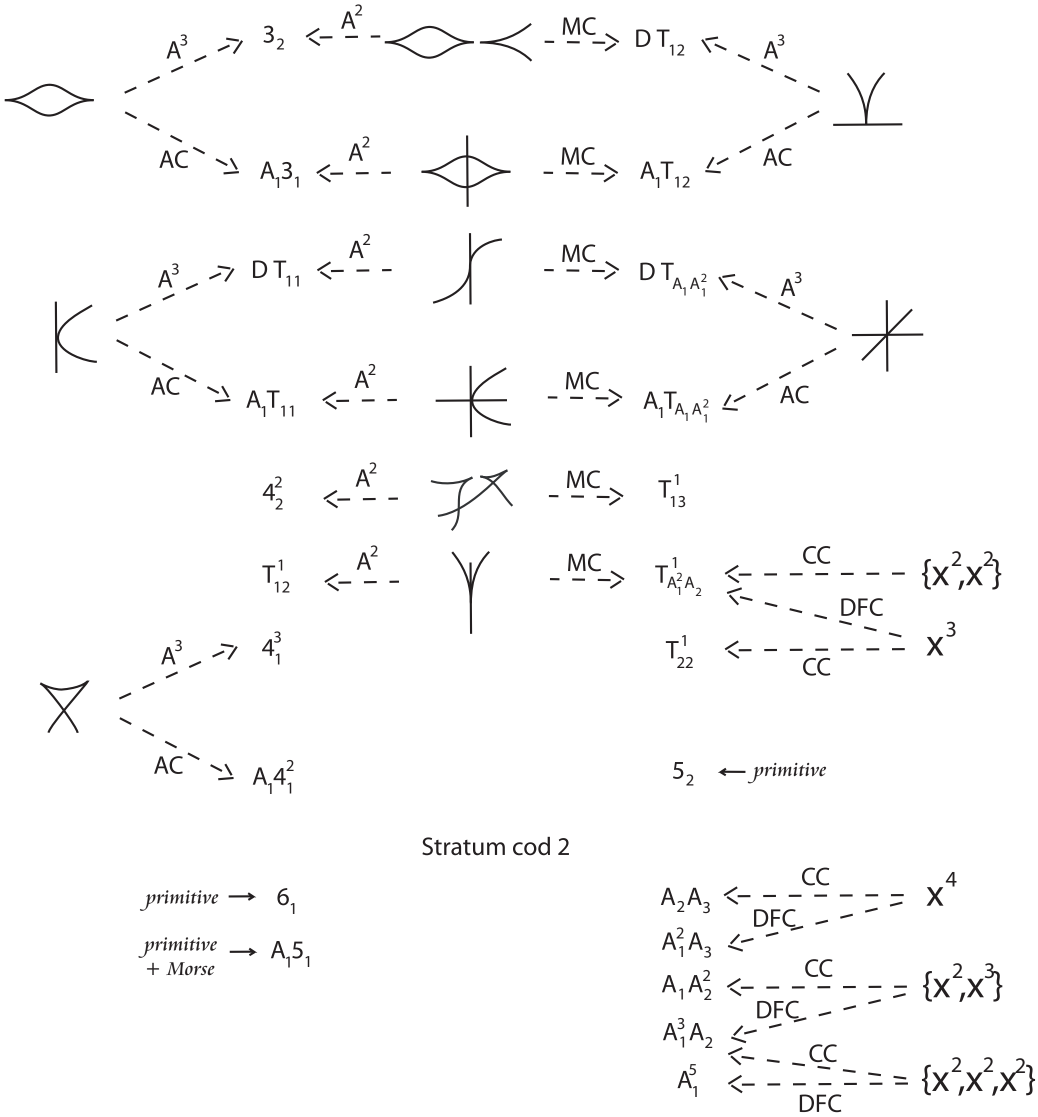

Figure 4.2: Codimension 2 germs and multigerms of maps from $\mathbb{C}^3$ to $\mathbb{C}^3$. CC and DFC stand for cuspidal concatenation and double fold concatenation respectively.

and its 1-parameter stable unfolding $\{(x^2+z, y, z), (x, y^3+xy, z)\}$. We can augment it by $\phi(z) = z^3$ or simultaneously augment and concatenate it to obtain the germs

$$\begin{cases} (x^2+z^3, y, z) \\ (x, y^3+xy, z) \end{cases} \quad \text{and} \quad \begin{cases} (x^2+z^2, y, z) \\ (x, y^3+xy, z) \\ (x, y, z^2) \end{cases} \quad \text{resp.}$$

However, the bigerm above can also be obtained by doing a monic concatenation to $(x, y^3 + x^3y)$, which would yield $\{(x, y^3 + x^3y + zy, z), (x, y, z^2)\}$ and is $\mathcal{A}$-equivalent to the normal form above. And the trigerm can be obtained by doing a monic concatenation to the bigerm $\{(x^2, y), (x, y^3 + x^2y)\}$, which would yield $\{(x^2, y, z), (x, y^3 + x^2y + yz, z), (x, y, z^2)\}$ and again is $\mathcal{A}$-equivalent to the normal form above.

Chapter 5

Simplicity of multigerms

A natural question arises regarding the simplicity of multigerms: Are the multigerms produced by all the operations mentioned up to now simple? If not, what conditions can be added in order to ensure simplicity? All of the results in this chapter can be found in [27].

For the simplicity of augmentations we have the following result for monogerms. We believe that it is true for multigerms too, but the proof in [27] can only be adapted for certain classes of multigerms.

Theorem 5.0.3. *Let $h : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ with $n \geq p-1$ be a non stable primitive monogerm which admits a 1-parameter stable unfolding H. Let g_1 and g_2 be augmenting functions and f_1 and f_2 the corresponding augmentations. Then*

$$f_1 \sim_{\mathcal{A}} f_2 \Rightarrow g_1 \sim_{\mathcal{K}} g_2.$$

This means that, given an augmentation $A_{H,g}(h)$ of a germ h, if the augmenting function g is not simple, then the augmentation is not simple.

Example 5.0.4.

i) The hypothesis of f being primitive is necessary. Let $f(x, y) = (x^3 + y^4 y, y)$ and $g(z) = z^4$, both of which are simple ($\mathcal{A}$ and $\mathcal{K}$, respectively). $A_{F,g}(f)(x, y, z) = (x^3 + (y^4 + z^4)x, y, z)$ is not simple because it is the same as $A_{H,\phi}(h)$ where $h(x) = x^3$ is primitive and $\phi(y, z) = y^4 + z^4$ is not simple.

ii) The converse is not true. Consider the primitive simple germ $f(z) = (z^2, z^5)$ and the simple function $g(x, y) = x^2+y^4$. The augmentation $A_{F,g}(f)(x, y, z) = (x, y, z^2, z^5 + (x^2 + y^4)z)$ is not simple (see [12]).

For the operation of simultaneous augmentation and concatenation we also have a very nice result:

Theorem 5.0.5. *Suppose $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ has a 1-parameter stable unfolding $F(x, \lambda) = (f_\lambda(x), \lambda)$. Let $g : (\mathbb{K}^p \times \mathbb{K}^{n-p+1}, 0) \to (\mathbb{K}^p \times \mathbb{K}, 0)$ be the fold map*

$(X,v) \mapsto (X, \Sigma_{j=p+1}^{n+1} v_j^2)$. *Suppose that* ϕ *is quasi-homogeneous and* $A_{F,\phi}(f)$ *is simple, then* $\mathcal{A}_e - cod(f) = 1$ *implies that* $\{A_{F,\phi}(f), g\}$ *is simple. Furthermore, if* g *is transverse to the limits of the tangent spaces of the strata of* $A_{F,\phi}(f)$*, then the converse is also true.*

Example 5.0.6. *i) Let* $f(y) = (y^2, y^3)$ *and consider the augmentations and concatenations*

$$\begin{cases} (y^2, y^3 + x^{k+1}y, x) \\ (y, x, 0) \end{cases} \tag{5.1}$$

These bigerms are called A_0S_k $(k \geq 1)$ *in [9] and [34] and are simple.*

ii) Let $f(y) = (y^2, y^5)$ *and consider the augmentation and concatenation*

$$\begin{cases} (y^2, y^5 + x^2y, x) \\ (y, x, 0) \end{cases} \tag{5.2}$$

The bigerm A_0B_2 *is not simple since* $\mathcal{A}_e - \mathrm{cod}(f) = 2$ *and the immersion is transverse to the strata of* B_2*. Therefore, the bigerms* A_0B_k *are not simple for* $k > 1$*.*

iii) The extra hypothesis for the converse of Theorem 5.0.5 to be true is necessary. If we simultaneously augment and concatenate the codimension 2 bigerm $\{(x^2, y), (x^2+y^3, y)\}$ *we obtain the codimension 4 simple trigerm ([34])*

$$\begin{cases} (x^2, y, z) \\ (x^2 + y^3 + z^2, y, z) \\ (x, y, z^2) \end{cases} \tag{5.3}$$

Notice that the double point curve for $\{(x^2, y, z), (x^2+y^3+z^2, y, z)\}$ *describes a cusp which is tangent in the limit to* g*.*

For the case of generalised concatenations many partial results can be obtained, but the operation is too general in order to get general results.

Theorem 5.0.7. *Let* $h = \{f, g\}$ *be a non-monic generalised concatenation (i.e.* g *is not a prism on a Morse function or an immersion) and suppose that* $\widetilde{\tau}(f) = \{0\}$*, then* h *is non simple.*

Example 5.0.8. *Let* $f(x) = x^4$ *and consider the 2-parameter stable unfolding* $F(x, \lambda_1, \lambda_2) = (x^4 + \lambda_1 x + \lambda_2 x^2, \lambda_1, \lambda_2)$*. We obtain the non simple cuspidal concatenation*

$$\begin{cases} (x^4 + yx + zx^2, y, z) \\ (x, y, z^3 + xy) \end{cases} \tag{5.4}$$

Proposition 5.0.9. *i) Let $f = \{f_1, \ldots, f_r\} : (\mathbb{K}^n, S) \to (\mathbb{K}^n, 0)$ be a primitive $\mathcal{A}_e$-codimension 1 germ, $n > 2$. Then the multigerm $h = \{f, A_1\}$ is not simple.*

ii) Let $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^{n+1}, 0)$ be a primitive $\mathcal{A}_e$-codimension 1 germ, $n > 3$. Then the multigerm $h = \{f, A_0\}$ is not simple.

Theorem 5.0.10. *Let $h = \{f, g\}$ is a multigerm with f a non stable germ and g a prism on a Morse function or an immersion and suppose that g is transverse to the limits of the tangent spaces of f. Then h is simple if and only if either f is an augmentation of an $\mathcal{A}_e$-codimension 1 germ (i.e. $h = \{A_{P,\phi}(p), g\}$ with $\mathcal{A}_e - \operatorname{cod}(p) = 1$) or h is one of the following:*

i) When $p = 1$, the bigerm of a Morse function and an A_2-singularity or the trigerm of 3 Morse functions.

ii) If $n = 1$ and $p = 2$, the codimension 2 bigerm $\{(x^2, x^3), (0, x)\}$, and if $n = p = 2$ the codimension 2 bigerm

$$\begin{cases} (x^4 + yx, y) \\ (x, y^2 + x) \end{cases} \tag{5.5}$$

and the trigerm

$$\begin{cases} (x^3 + xy, x) \\ (x, y^2) \\ (x, y^2 + x) \end{cases} \tag{5.6}$$

iii) When $(n, p) = (3, 4)$, the bigerm

$$\begin{cases} (u, v, x^3 + ux, x^4 + vx) \\ (u, u, v, x) \end{cases} \tag{5.7}$$

Chapter 6

Future research

The reader may have noticed that in many of the examples, in order to carry out the computations one must know the liftable vector fields of certain germs. For example, given a bigerm $h = \{f, g\}$, it is proved in [26] that the following sequence is exact

$$0 \longrightarrow \frac{\theta(g)}{tg(\theta_{n+1})+wg(Lift(f))} \longrightarrow N\mathcal{A}_e(\{f,g\}) \xrightarrow{\overline{h}} N\mathcal{A}_e(f) \longrightarrow 0$$

Therefore

$$\mathcal{A}_e - cod(\{f,g\}) = \mathcal{A}_e - cod(f) + dim_{\mathbb{K}} \frac{\theta(g)}{tg(\theta_{n+1}) + wg(Lift(f))}.$$

However, computing $Lift(f)$ is not easy in general. The study of liftable vector fields is still an active field of research. Recently, in [21], a systematic method to obtain liftable vector fields of some corank 1 germs is given. In particular the method can be applied to stable germs. Furthermore, in [22], it is shown how to recover liftable vector fields of any germ from the liftable vector fields of an s-parameter stable unfolding. This is an important starting point but there is still much to do. For example, little is known about liftable vector fields of corank 2 maps.

Very little is known of corank 2 germs in general. [29] and [1] are recent developments in this area. It would be interesting to study how the operations studied throughout the paper behave in corank 2, since they may provide many examples of finitely determined corank 2 germs.

There is still much to be understood about the simplicity of multigerms. Some advances are being done in [**?**], but there is much more to be done.

In certain contexts such as applications of Singularity Theory to Differential Geometry, classifications with geometric subgroups different from $\mathcal{A}$ are carried out. It would be interesting to explore how all these operations and classification techniques apply to ${}_V\mathcal{K}$ and ${}_V\mathcal{R}$ classifications.

Many of the operations depend on the existence of a 1-parameter stable unfolding. However, it is not fully understood when a germ admits a 1-parameter stable unfolding. Certainly there is a bound on the multiplicity, but not all germs with multiplicity lower than this bound admit 1-parameter stable unfoldings. Certain evidence shows that most simple germs admit 1-parameter stable unfoldings and a list of exceptions can be produced, but this is still far from understanding what condition ensure the existence. Maybe it could be related to the liftable vector fields of the germ.

The Mond conjecture is known to be satisfied by augmentations and codimension 1 and 2 monic and binary concatenations, and so, in [6] it is concluded that all codimension 1 germs satisfy the Mond conjecture. In [5] is is proved that the simultaneous augmentation and concatenation satisfies the Mond conjecture. Due to Theorem 4.4.2, if the Mond conjecture is satisfied for multigerms obtained via the operation of generalised concatenation, then all codimension 2 germs would satisfy the conjecture. It may be difficult to prove this in general, but a possibility is to start with particular examples of non-monic generalised concatenations. This could be an alternative method to prove the Mond conjecture.

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