THE SPACE OF ARCS AND THE NASH PROBLEM.

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The course will start out with the definition of algebraic varieties over a field and the notions of singularities and their resolutions. Given an algebraic variety X, its **resolution of singularities** is a birational proper morphism $X' \to X$ such that X' is non-singular.

The early nineteen sixties were marked by a major breakthrough in algebraic geometry: H. Hironaka solved a long standing problem by proving that every algebraic variety over a field of characteristic zero admits a resolution of singularities [4]. This result provided an inspiration to John Nash for several extremely fruitiful ideas, one of the most important of them being the definition of the space of arcs of an algebraic variety X and the formulation of the Nash problem [7].

Let X be an algebraic variety over a ground field k. An **arc** on X is, by definition, a morphism $Spec\ k[[t]] \to X$. The curve $Spec\ k[[t]]$ should be thought of as a germ of a test curve. As a topological space, it only has two points, the special one (the origin) and the generic one. Its analogue in complex analysis is the germ of the unit disc at the origin. The image of $Spec\ k[[t]]$ in X is a parametrized formal curve, contained in X.

The space of arcs on X has a natural structure of an infinite-dimensional algebraic variety. For simplicity, assume that X has one isolated singular point ξ (in the course we will study the general case of arbitrary singularities). Let $\mathcal{H}$ denote the space of all arcs on X whose image in X passes through ξ. Let $\pi : X' \to X$ be a resolution of singularities of X. The inverse image $E := \pi^{-1}(\xi)$ of the singular point in X' is called the **exceptional set** of π. Let

$$E = \bigcup_{i=1}^{r} E_i \tag{1}$$

denote the decomposition of E into irreducible components. In a preprint dating to 1965 (published in 1995 [7]) Nash proved that the decomposition (1) induces a decomposition $\mathcal{H} = \bigcup\limits_{i=1}^{r} \mathcal{H}_i$ of $\mathcal{H}$ where each $\mathcal{H}_i$ is an infinite-dimensional irreducible algebraic subvariety of $\mathcal{H}$. For irreducible components E_i of E Nash defined the notion of being **essential**. He conjectured that the varieties $\mathcal{H}_i$ corresponding to the essential components E_i are precisely the irreducible components of $\mathcal{H}$.

After studying several examples of the Nash correspondence, mainly for varieties X of dimension two, we will discuss the affirmative solution of the two-dimensional Nash problem ([3] and [1]) as well as the current state of knowledge for X of dimension three or higher ([1], [2], [5] and[6]).

References

1. T. de Fernex and R. Docampo, *Terminal valuations and the Nash problem*, DOI 10.1007/s00222-015-0597-5, arXiv:1404.0762 [math.AG], Invent. Math. (2015), 1-29.
2. T. de Fernex, *Three-dimensional counter-examples to the Nash problem*, Compositio Math. **149** (2013), 1519-1534.
3. J. Fernandez de Bobadilla and M. Pe, *The Nash problem for surfaces*, Ann. Math. **176, Issue 3** (2012), 2003-2029..
4. H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Ann. Math. **79** (1964), 109–326.
5. J. Kollár and S. Ishii, *The Nash problem on arc families of singularities*, Duke Math. J. **120, Number 3** (2003), 601-620.
6. J. Johnson and J. Kollár, *Arc spaces of cA-type singularities*, 1306.1208.
7. J. F. Nash, *Arc structure of singularities*, DOI: 10.1215/S0012-7094-95-08103-4, Duke Math. J. **81** (1995), 31-38.