My long lived conjecture

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0.1 Introductory note – please read!

This minicourse is aimed at student and researchers who have some background in singularity theory but have not studied singularities of mappings in depth. It makes use of some moderately sophisticated ideas from commutative algebra, of which it does not assume prior knowledge. I hope that by seeing how the commutative algebra works to prove significant results on singularities of mappings, students will gain an appreciation for its meaning and power. The key notion is the depth of module over a commutative ring (see the definition in the Appendix, Section 5). This will appear at several points. More elementary is the notion of (Krull) dimension. This is defined in commutative algebra in a way that makes no apparent reference to any geometry. Fortunately, the commutative rings we use are always rings of germs of complex analytic functions in m complex variables, $\mathcal{O}_m$, and for a module M over $\mathcal{O}_m$, its dimension is the (complex) geometric dimension of its support, the variety of zeros in $\mathbb{C}^m$ of its annihilator ideal. Depth is always less than or equal to dimension, and modules for which the two are equal are dignified with a special name; they are *Cohen-Macaulay* modules. They are the "best-behaved" modules. Cohen-Macaulay modules play a special role in proving "conservation of multiplicity", the phenomenon, which is common in analytic geometry, whose most familiar example may be the way that an isolated critical point of a function $g \in \mathcal{O}_m$ breaks up, under deformation, into a collection of isolated critical points whose multiplicities (Milnor numbers) add up to the multiplicity of the original critical point. This example is relevant here. By showing that a critical point with Milnor number $\mu < \infty$ breaks up into μ non-degenerate critical points (i.e. with Milnor number 1), we are able to describe the topology of the Milnor fibre, and relate a topological property, the rank of its middle homology, to an algebraic property – the complex vector space dimension of the jacobian algebra $\mathcal{O}_m / J_g$, which at first seems to belong to deformation theory. The main thrust of this minicourse is exactly the same. We will spend quite a lot of effort showing that certain modules are Cohen-Macaulay in order to deduce conservation of multiplicity and conclude that some number, defined as the vector space dimension of a certain module, actually tells us about the topology of a stable perturbation of a map-germ with isolated instability.

The lectures will cover the same ground as these lecture notes. They will work best if you have these notes with you at the lectures. This will enable me not to have to write everything on the board, and adjust the lecture to respond to the interests and questions of participants without needing to cover everything that is in the lecture notes.

There are some exercises, and of course it will help if you try to do them during the minicourse. Be confident that success is less important than grappling with the ideas.

Lecture 1

1 Basic definitions, and some examples

1.1 μ versus τ

The conjecture concerns two numbers with which on can measure the complexity or degeneracy of an analytic map-germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ with isolated instability. At the outset we declare two such germs to be *left-right equivalent* (or $\mathscr{A}$*-equivalent*) if their expression as $|S|$-tuples of power series are the same with respect to suitable coordinate systems on source and target. Note that the notion of germ of mapping is particularly straightforward in analytic geometry – a germ is determined by a power series, and vice versa. Slightly trickier is the notion of *good representative* of a germ. We will come on to that later.

The first number its left-right codimension, or $\mathscr{A}_e$-codimension – briefly, the number of parameters needed for a versal unfolding of f, (i.e. which explores all possible deformations, up to left-right equivalence). Equivalently, this is the minimum dimension of the parameter space of a parametrised family of maps in which such a singularity (i.e. a germ in the given $\mathscr{A}$-equivalence class) may occur unavoidably.

The $\mathscr{A}_e$-codimension be calculated algebraically by a variety of methods, which I will describe shortly. It is finite if and only if the germ has "isolated instability", just as for a hypersurface singularity the Milnor number μ is finite if and only if the singularity is isolated.

The second number is a measurement of the topological complexity of the image of a stable perturbation of a good representative of the germ. To make this notion precise requires some careful definitions, so for now let me just say briefly that given a singularity, we pick a "good representative", on a "sufficiently small neighbourhood" (within which the mapping is topologically a cone on its boundary), and then perturb the representative so that within this neighbourhood, the instability disappears. We need to assume that such a perturbation exists. For this we require $(n, n+1)$ to be in Mather's "nice dimensions" ([Mat71]), where every map germ with isolated instability has a stable perturbation – for $(n, n+1)$, this turns out to mean that $n \leq 15$ – or, alternatively, that f lies within a class, such as corank 1 map-germs, or multi-germs of immersions, where, again, such perturbations always exist. Now, it turns out that the image of a stable perturbation has the homotopy type of a wedge of spheres of dimension n (the topological dimension is $2n$, of course, so this is "middle dimension"), and the number we are after is the number of spheres in this wedge. It is called the *image Milnor number*, μ_I. **Caution:** this is not the Milnor number of the image. If $n > 1$ then the image of any germ other than an immersion has non-isolated singularities, and its Milnor number is infinite. The image Milnor number refers to the topology of the image of a stable perturbation, not the topology of a smoothing of the image.

The conjectural relation between the two numbers, is simply this

Conjecture 1.1.

$$\mu_I(f) \geq \mathscr{A}_e - codimension(f),$$

with equality in case f is quasihomogeneous.

("quasihomogeneous" means weighted homogeneous with respect to suitable coordinates). The conjectured relation betwen μ_I and $\mathscr{A}_e$-codimension is closely analogous to the well known relation $\mu \geq \tau$ for isolated hypersurface singualrities and isolated complete intersection singularities, where μ is the Milnor number, the rank of the middle homology of a nearby non-singular level set, which also becomes an equality in the quasihomogeneous case, and is sometimes referred to as a "μ–τ type" relation.

Before I give any greater precision, let me give an example with which Terry Wall sparked my interest many years ago. Some of the terms in this informal discussion will be defined formally in the next section. We look at a smoothly bent and twisted piece of wire, ideally knotted so as to guarantee a complicated picture. Closing one eye, what we see is in effect a planar projection of the wire. The global picture is made up of a number of local pictures – by which I mean, pictures which are homeomorphic to a cone on their boundary. This is true of any line drawing – or any semi-algebraic set – and is something we are intuitively familiar with.

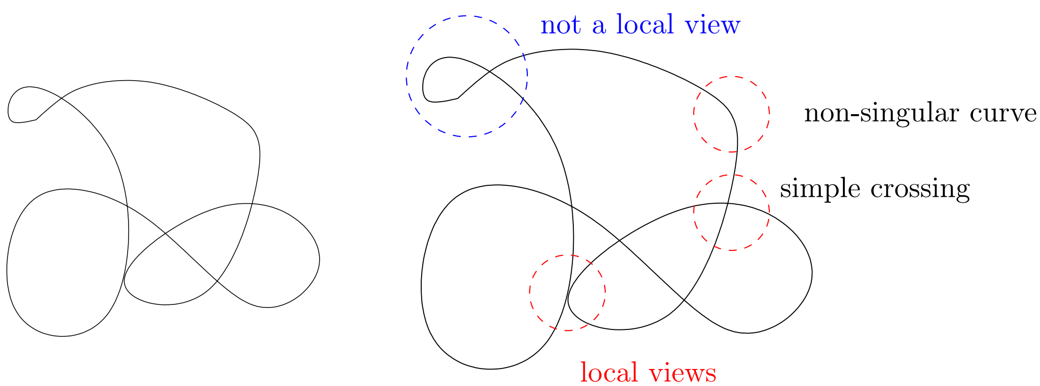
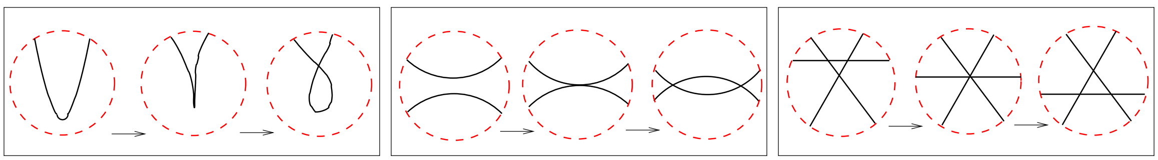


Figure 1: Local and non-local views

We will refer to any convex open set within which the picture is homeomorphic to a cone on its boundary as a *Milnor ball.*

Each diffeomorphism class of local picture centred at a point P in the plane corresponds to a left-right equivalence class of multi-germ $(\mathbb{R}, S) \to (\mathbb{R}^2, P)$ parameterising it. To each of these classes we can associate its *viewing set*: the set of centres of projection ("viewpoints") $Q \in \mathbb{R}^3$ such that the planar projection from Q exhibits a local picture of this class. For example, for the first order cusp, which one sees by looking at the curve along a tangent line, the viewing set is the tangent developable surface of the curve. For the triple point, the viewing set is the collection of trisecant lines.

Two of these local views are stable: a portion of non-singular curve, and a simple crossing. Up to diffeomorphism, they are not changed by slight movements of the viewpoint. The others are all unstable, with each having a single unstable point. For a suitably general bent wire, each local view, with one exception, will be versally unfolded by moving the centre of projection. In this case, any transversal to the viewing set becomes the base of a miniversal unfolding, and thus the dimension of this transversal – i.e. the codimension of the viewing set – is equal to the A_e – codimension of its parameterisation. (The exception is the quadruple point, whose cross ratio is not changed by moving Q along the quadrisecant line.) There are three whose codimension is 1: the cusp, the tacnode and the triple point. Their viewing sets, or more precisely the closure of their viewing sets, separate the ambient $\mathbb{R}^3$.



RI RII RIII

Figure 2: The three Reidemeister moves – codimension 1 local views of a generic space curve

To pass from one knot projection to another, the centre of projection Q must pass through them. This is essentially why they appear in knot theory, as the three Reidemeister moves.

By moving the centre of projection Q, we can perturb any given local view so that it becomes stable. Within the original Milnor ball, the only singular points of the curve are simple crossings, which emerge from the centre as we shift our viewpoint. Like any planar graph, the part of the curve inside the Milnor ball is homotopy equivalent to a wedge of circles. Different stable perturbations may have different numbers of circles, but, it happens that for every one of the singularities that is versally unfolded in the family of projections, *the maximal number of circles in the wedge is equal to the codimension of the viewing set.* This is an instance of the equality in Conjecture 1.1. You can see the 1-cycle on the right in the pictures of RI and RII, and on both sides in RIII.

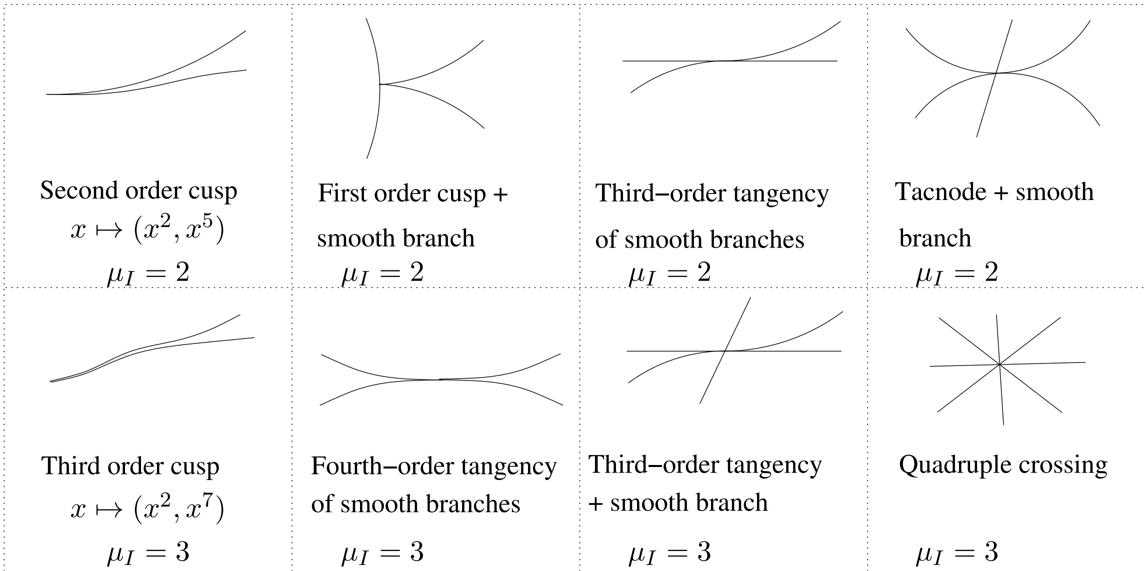


Figure 3: higher codimension local views of a generic space curve

The number of circles in a maximal wedge for the quadruple point, three, is one too many – the viewing set is made up of lines which cut the curve four times, and so has codimension 2 – although it is quasihomogeneous.

Exercise 1.2. Find maximal stable perturbations of each of the germs in Figure 3, and check that in each case the image is homotopy-equivalent to a wedge of μ_I circles. Note that when the real picture has the same homology as the complex, then the inclusion of one in the other is a homotopy equivalence.

1.2 Slightly more technical detail

The most straightforward way of defining the A_e − codimension of a germ $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) is as the vector space dimension of

$$\frac{\{\frac{d}{dt} f_t|_{t=0} : f_0 = f\}}{\{\frac{d}{dt}\left(\psi_t \circ f \circ \varphi_t\right)|_{t=0} \ : \ \psi_0 = \mathrm{id}_{\mathbb{K}^p}, \varphi_0 = \mathrm{id}_{\mathbb{K}^n}\}} \tag{1.1}$$

Both numerator and denominator here can be expressed more explicitly. Clearly, for each point x in the domain,

$$\frac{d}{dt} f_t(x)|_{t=0} \in T_{f(x)}\mathbb{K}^p.$$

Thus $x \mapsto \frac{d}{dt} f_t(x)|_{t=0}$ is a map from $(\mathbb{K}^n, S) \to T\mathbb{K}^p$ "over f": it gives the diagonal arrow in a commutative diagram

$$\begin{array}{ccc} T\mathbb{K}^n & \xrightarrow{df} & T\mathbb{K}^p \\ \downarrow & \nearrow & \downarrow \\ \mathbb{K}^n & \xrightarrow[f]{} & \mathbb{K}^p \end{array} \tag{1.2}$$

in which the vertical maps are bundle projections. If $\hat{f} : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ is *any* map germ, then

$$f_t(x) = f(x) + t\hat{f}(x)$$

is a 1-parameter deformation whose derivative is $\hat{f}$. Thus the numerator in (1.1) is the free $\mathcal{O}_{\mathbb{K}^n,S}$ module on generators $\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_p}$. We denote it by $\theta(f)$.

In particular, the expressions $\frac{\partial \varphi_t}{\partial t}|_{t=0}$ and $\frac{\partial \psi_t}{\partial t}|_{t=0}$, in the denominator of (1.1), define germs of vector fields on $(\mathbb{K}^n, S)$ and $(\mathbb{K}^p, 0)$ respectively. Denoting these by ξ and η we have

$$\frac{d(\psi_t \circ f \circ \psi_t)}{dt}|_{t=0} = df \circ \xi + \eta \circ f.$$

Once again, every germ of vector field ξ and η can appear in this way, so the denominator in (1.1) is equal to

$$\{df \circ \xi : \xi \in \theta_{\mathbb{K}^n,S}\} + \{\eta \circ f : \eta \in \theta_{\mathbb{K}^p,0}\}$$

We write the operators $\xi \mapsto df \circ \xi$ and $\eta \mapsto \eta \circ f$ as tf and ωf respectively, so finally the denominator in (1.1) takes the form

$$tf(\theta_{\mathbb{K}^n,S}) + \omega f(\theta_{\mathbb{K}^p,0}).$$

We call it the *extended tangent space* to the orbit of f, and denote it by $T\mathcal{A}_e f$.The quotient

$$\frac{\theta(f)}{tf(\theta_n) + \omega f(\theta_p)} \tag{1.3}$$

is denoted by $T^1_{\mathscr{A}_e} f$. Writing this in terms of generators, this becomes

$$\frac{\mathcal{O}_n\{\partial/\partial y_1, \ldots, \partial/\partial y_p\}}{\mathcal{O}_n\{\partial f/\partial x_1, \ldots, \partial f/\partial x_n\} + f^*(\mathcal{O}_p\{\partial/\partial y_1, \ldots, \partial/\partial y_p\})} \tag{1.4}$$

The $\mathcal{A}_e$-codimension of f is the complex vector space dimension of (1.3). If this dimension is 0 then f is "infinitesimally stable"; in fact from this it follows, by Martinet's versality theorem (1.6 below) that f is stable: every unfolding is trivial.

If $F : (\mathbb{C}^n \times \mathbb{C}^d, S \times 0) \to (\mathbb{C}^p \times \mathbb{C}^d, (0,0))$, with $F(x,u) = (\tilde{f}(x,u), u)$, is an unfolding of f, there is a "relative version" of $T^1_{\mathscr{A}_e} f$, defined by parameterising all the ingredients of the quotient (1.4):

$$T^{1\ \mathrm{rel}}_{\mathscr{A}_e} F := \frac{\mathcal{O}_{n+d}\{\partial/\partial y_1, \ldots, \partial/\partial y_p\}}{\mathcal{O}_{n+d}\{\partial \tilde{f}/\partial x_1, \ldots, \partial \tilde{f}/\partial x_n\} + F^*\big(\mathcal{O}_{p+d}\{\partial/\partial y_1, \ldots, \partial/\partial y_p\}\big)} \tag{1.5}$$

also written as

$$\frac{\theta(F/\mathbb{C}^d)}{t\tilde{f}(\theta_{n+d/d}) + \omega\tilde{f}(\theta_{p+d/d})}$$

Exercise 1.3. *Check that reducing* $T^1_{\mathscr{A}_e}{}^{rel}F$ *module* $\mathfrak{m}_d$ *gives* $T^1_{\mathscr{A}_e}f$*, i.e.*

$$T^1_{\mathscr{A}_e}{}^{rel}F \otimes_{\mathcal{O}\,p+d} \frac{\mathcal{O}_d}{\mathfrak{m}_d} = \frac{T^1_{\mathscr{A}_e}{}^{rel}F}{(u_1,\ldots,u_d)T^1_{\mathscr{A}_e}{}^{rel}F} = T^1_{\mathscr{A}_e}f.$$

Example 1.4. *The calculations here are purely formal, and pay no attention to the convergence of power series. However, thanks to general approximation theorems which I will not go into, their conclusions are correct.*

(1) The germ in the centre of the first Reidemeister move can be parametrised by $f(x) = (x^2, x^3)$*. Every power of* x*, except for* x *itself, can be written as a monomial in* x^2 *and* x^3*, so*

$$\omega f(\theta_{\mathbb{K}^2,0}) + Sp_{\mathbb{K}}\left\{\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x \end{pmatrix}\right\} = \theta(f).$$

Now $\begin{pmatrix} 0 \\ x \end{pmatrix}$ *is not in* $T\mathcal{A}_e f$*, since the order of the coefficient of* $\partial/\partial y_2$ *in every member of* $T\mathcal{A}_e f$ *is at least* 2*. On the other hand,*

$$tf\left(\frac{\partial}{\partial x}\right) = \begin{pmatrix} 2x \\ 3x^2 \end{pmatrix}$$

and it follows that only $\begin{pmatrix} 0 \\ x \end{pmatrix}$ *is missing from* $T\mathscr{A}_e f$*, and*

$$T\mathcal{A}_e f + Sp_{\mathbb{K}}\left\{\begin{pmatrix} 0 \\ x \end{pmatrix}\right\} = \theta(f) \tag{1.6}$$

and f *has* $\mathcal{A}_e$*-codimension 1.*

(2) For a multi-germ $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ *with* $S = \{s_1, \ldots, s_k\}$*, we denote by* f_j*, for* $j = 1, \ldots, k$*, the associated mono-germs* $(\mathbb{K}^n, s_j) \to (\mathbb{K}^p, 0)$*. Elements of* $\theta(f)$ *can be represented by* $p \times k$ *matrices, with the* j*'th column representing the elements of* $\theta(f_j)$*. For example, consider the bi-germ*

$$f : \begin{cases} s \mapsto & (s, 0) \\ t \mapsto & (0, t) \end{cases}$$

parameterising a transverse crossing of two immersed branches. It is infinitesimally stable. To see this, observe that if a, b, c *and* d *all vanish at* 0 *then the element*

$$\begin{pmatrix} a(s) & c(t) \\ b(s) & d(t) \end{pmatrix} \tag{1.7}$$

of $\theta(f)$ is equal to

$$\omega f\begin{pmatrix}(y_1)+c(y_2)\\ b(y_1)+d(y_2)\end{pmatrix},$$

while if a_0, b_0, c_0, d_0 are arbitrary constants then

$$tf(a_0-c_0, d_0-b_0)+\omega f\begin{pmatrix}c_0\\ b_0\end{pmatrix}=\begin{pmatrix}a_0 & c_0\\ b_0 & d_0\end{pmatrix}.$$

This completes the proof of infinitesimal stability.

(3) *Consider the perturbation $f_t : x \mapsto (x^2, x^3 - tx)$ of the germ f in Example (1) above; it is an immersion, and for real $t > 0$, or any complex $t \neq 0$, it has one double point – the points $\pm\sqrt{t}$ have the same image, $(t, 0)$. The two branches of the image meet transversely at $(t, 0)$, and otherwise f_t is an embedding. Thus it is a stable perturbation of f. The image has the homotopy type of a circle, as you can see in the right hand column of Figure 2.*

Similar slightly more complicated calculations show that the codimension of Reidemeister moves II and III is also 1, again equal to the rank of their vanishing homology.

Exercise 1.5. (i) Check this. (ii) Find $\mathscr{A}_e$-codim f when $f(x) = (x^2, x^5)$.

Before going on, I point out that the reason that the $s\mathcal{A}_e$ codim of f is the number of parameters needed for a miniversal unfolding of f is the following *versality theorem*, proved by Jean Martinet in [Mar77] (and more accessibly published in [Mar82]). Here $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$.

Theorem 1.6. *An unfolding $F : (\mathbb{K}^n \times \mathbb{K}^d, S \times \{0\}) \to (\mathbb{K}^p \times \mathbb{K}^d, (0,0))$ of $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$, ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), $F(x, t_1, \ldots, t_d) = (f_t(x), u)$, is $\mathcal{A}_e$-versal if and only if the images in $\theta(f)/T\mathcal{A}_e f$ of the initial velocities $\partial f_t/\partial t_i|_{t=0}, i = 1, \ldots, d$, span it as a $\mathbb{K}$-vector space.*

Versality of $F(x,t) = (f_t(x), t)$ means that every unfolding $G(x, u)$ of f is parameterised-equivalent to an unfolding induced from F by a map of parameters $u \mapsto t(u)$. It follows that every perturbation of f is equivalent to f_t for some t.

Note that from the versality theorem it follows that if f is infinitesimally stable then it is stable.

A versal unfolding contains every possible perturbation of f, up to equivalence; if f has a stable perturbation at all, then the set of parameters t of parameter values t for which f_t is stable is the complement of an analytic subset, the *bifurcation set.* In Mather's nice dimensions, the bifurcation set is a proper analytic subset of the parameter space of the unfolding. In the complex case this subset does not separate the parameter space, so any two good parameter values t_0 and t_1 can be joined by a path avoiding the bifurcation set. From this it follows that f_{t_0} and f_{t_1} are topologically equivalent, thus proving the (topological) uniqueness of the stable perturbation over $\mathbb{C}$.

There are five "Reidemeister moves" – A_e – codimension 1 map-germs - for mappings from 2-space to 3-space. They were first described by Victor Goryunov in [Gor91]. I list them here, and in each case describe a 1-parameter versal unfolding, which the reader can check by finding a basis for $\theta(f)/T\mathcal{A}_e f$ and applying Theorem 1.6. They are

(1) The S_1 singularity (birth of two Whitney umbrellas), parameterised by

$$(x,y) \mapsto (x, y^2, y^3 \pm x^2 y).$$

Here, as in (2), the two forms, distinguished by $\pm$ in the third component, are inequivalent over $\mathbb{R}$ but equivalent over $\mathbb{C}$. The unfolding $F(x,y,t) = (f_t(x,y), t)$, with $f_t(x,y) = (x, y^2, y^3 \pm x^2 y + ty)$, is $\mathcal{A}_e$- versal.

(2) The Morse tangency (the surface equivalent of the tacnode RII), a bi-germ parameterised by

$$\begin{cases} (x_1, y_1) & \mapsto & (x_1, y_1, 0) \\ (x_2, y_2) & \mapsto & (x_2, y_2, x_2^2 \pm y_2^2) \end{cases}$$

A versal unfolding on parameter u is obtained by adding the unfolding parameter t to the third component of f_1 (or of f_2).

(3) The degenerate triple point, parameterised by

$$\begin{cases} (x_1, y_1) & \mapsto & (x_1, y_1, 0) \\ (x_2, y_2) & \mapsto & (0, x_2, y_2) \\ (x_3, y_3) & \mapsto & (x_3 - y_3^2, y_3, -x_3 - y_3^2) \end{cases}$$

Here three immersed surfaces meet two-by-two transversely, with each tangent to the curve of intersection of the other two. The unfolding in which f_3 is modified to $f_{3,t}(x_3, y_3) = (x_3 - y_3^2 + t, y_3, -x_3 - y_3^2 + t)$ is $\mathcal{A}_e$-versal.

(4) The umbrella with an immersed plane passing through it, parameterised by

$$\begin{cases} (x_1, y_1) & \mapsto & (x_1, y_1^2, x_1 y_1) \\ (x_2, y_2) & \mapsto & (x_2, -x_2, y_2) \end{cases}$$

A versal unfolding is obtained by adding t to the second component of f_2.

(5) The quadruple point, in which four immersed planes meet, with each three in general position. The three coordinate planes and a fourth plane with equation $u + v + w = 0$ can be parameterised by

$$\begin{cases} (x_1, y_1) & \mapsto & (0, x_1, y_1) \\ (x_2, y_2) & \mapsto & (x_2, 0, y_2) \\ (x_3, y_3) & \mapsto & (x_3, y_3, 0) \\ (x_4, y_4) & \mapsto & (x_4, y_4, -x_4 - y_4) \end{cases}$$

This is versally unfolded by adding (t, t, t) to f_4.

As Goryunov's drawings show, each one (taking the positive variant in the first and second case, where there is a choice of sign) can be perturbed to a mapping whose image is a homotopy 2-sphere.

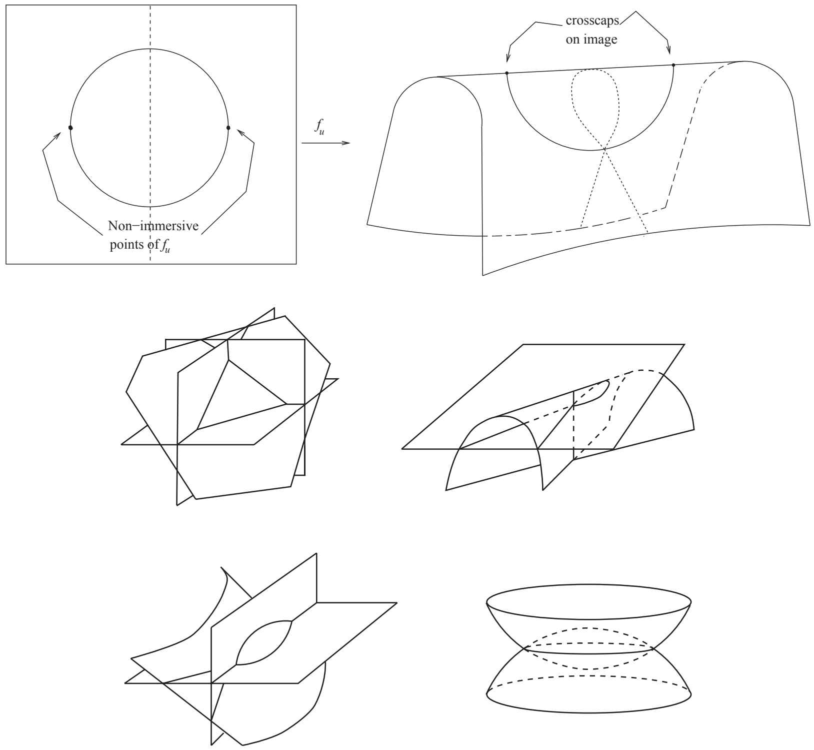


Figure 4: Images of stable perturbations of the five $\mathscr{A}_e$-codimension 1 singularities of maps from surfaces to 3-space.

Let g_t be a defining equation of one of the images in Figure 4, obtained by perturbing the defining equation g_0 of the corresponding $\mathscr{A}_e$–codimension 1 germ. The image, $g_t^{-1}(0)$, encloses a chamber. Because the chamber is compact, g_t must have a local maximum or minimum in its interior. In the stable images you found in Exercise 1.2, the plane curve encloses μ_I plane chambers, and inside each one, once again, there is a local maximum or minimum. It turns out that in each case there is precisely one critical point in the interior of each chamber, and it is non-degenerate, though this is not obvious. An extension of this crucial fact to the complex domain gives us the means to determine the value of $\mu_I(f)$:

Theorem 1.7. *(*D. Siersma, [Sie91]*) Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be $\mathscr{A}$-finite. Then $\mu_I(f)$ is equal to the sum of the Milnor numbers of the critical points of g_t which move off the image $hg_t^{-1}(0)$ as the parameter t moves away from 0. In particular, if all the critical points are non-degenerate, then $\mu_I =$ number of critical points.* □

The proof shows that up to homotopy, each non-degenerate critical point contributes an n-sphere to the image of a stable perturbation of f. The argument is just Morse theory, and uses the fact that a non-degenerate critical point of a holomorphic function g is also a non-degenerate critical point of its modulus $|g|$, which is therefore a (real) Morse point in the usual sense. The index of this critical point is always $n+1$ (i.e. one half of the real dimension of the ambient space), so passing the critical value c_j of $|g_t|$ glues in, to $|g_t|^{-1}([0, c_j))$, an $n+1$-cell whose boundary is the n-sphere. The argument applies to any deformation of a hypersurface singularity in which nothing special crosses the boundary of the Milnor ball as the parameter moves away from 0. This is ensured by choosing $0 < |t|$ sufficiently small that during the deformation the hypersurface is always transverse to the boundary of the ball.

In the real case, the index of a critical point which move off the image or discriminant may be different from the ambient dimension. For example, in the drawing in Figure 2 of Reidemeister move RII, in the stable perturbation shown on the left the critical point has index 1, and the vanishing cycle is a 0-sphere – the image now has two contractible path components.

Exercise 1.8. Where is the vanishing cycle on the left in Reidemeister 1?

Theorem 1.7 also applies to the discriminant (set of critical values) of $\mathscr{A}$-finite map-germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ with $n \geq p$. The discriminant (set of critical values) of a stable perturbation once again has the homotopy type of a wedge of spheres in middle dimension, in this case $p-1$, whose number, $\mu_\Delta(f)$, is once again the sum of the Milnor numbers of the isolated critical points which move off the discriminant as the parameter moves away from 0. In the next two lectures we will look at discriminants alongside images, since the arguments are in most respects the same. However in an important respect discriminants are easier to deal with than images, and the result corresponding to the conjecture was proved in 1991:

Theorem 1.9. [DM91] *For an $\mathscr{A}$-finite map-germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ with $n \geq p$, and (n,p) nice dimensions, $\mu_\Delta(f) \geq \mathscr{A}_e - codimension(f)$, with equality if f is quasihomogeneous.* □

We will sketch the proof of this in the next lectures, because the same argument nearly works to prove the conjecture, and indeed does work in the case of multi-germs of immersions:

Theorem 1.10. *For an $\mathscr{A}$-finite map-germ of immersions, $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$. $\mu_I(f) \geq \mathscr{A}_e - codimension(f)$, with equality if f is quasihomogeneous.* □

Note there is no restriction that $(n, n+1)$ should be nice dimensions here.

The general case of the conjecture would be proved if we could plug one crucial gap.

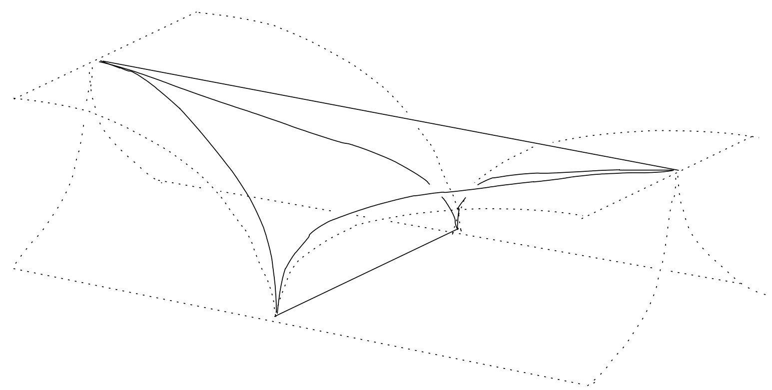


Figure 5: Discriminant of a stable perturbation of an $\mathscr{A}_e$ codimension 1 bi-germ, in which each component mono-germ is a trivial unfolding of a Whitney cusp mapping. The discriminant of each monogerm is drawn with dotted lines. Each has a (straight) cuspidal edge, drawn with a solid line.The (curved) intersection of the two discriminants is also drawn with a solid line. The union of the two discriminants is homotopy-equivalent to the curvilinear tetrahedron outlined by the solid lines, which is, once again, a homotopy 2-sphere,

1.3 Evidence for the conjecture

(1) Duco van Straten and Theo de Jong proved it in case $n = 2$, in [dJvS91], and I found a second proof ([Mon91]) for the same case, based on their idea of the homomorphism of Theorem 2.1 below. A variant of this method was used to prove it for the case $n = 1$ ([Mon95]).

(2) Kevin Houston gave an elegant proof in the case of map-germs of multiplicity 2, in [Hou98]. All such germs have corank 1, and for a corank 1 germ, the image Milnor number can be calculated using the alternating homology of the multiple point spaces $D^k(f)$. For a germ f of multiplicity 2, only $D^2(f)$ is non-empty, and Houston showed a direct relation between the rank of its alternating homology and $T^1_{\mathscr{A}_e} f$.

(3) Kevin Houston also showed that if $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ satisfies the conjecture, and has a 1-parameter stable unfolding $F(x, u) = (\tilde{f}_(x, u), u)$, and if $g \in \mathcal{O}_s$ defines an isolated hypersurface singularity then under mild additional hypotheses the "augmentation" $A_{F,g}$ of f defined by $A_{F,g}(x, u) = (\tilde{f}(x, g(v)), v)$ also satisfies it. His theorem has recently been strengthened by Raul Oset and Ignacio Breves.

(4) Kevin Houston and Neil Kirk classified simple singularities of maps $\mathbb{C}^3 \to \mathbb{C}^4$, [HK99], and verified the conjecture for all of them.

(5) Juan-José Nuño Ballesteros and Roberto Giménez Conejero ([GCNB23]) proved a weak version of the conjecture: if $\mathscr{A}_e$-codimension $(f) > 0$ then $\mu_I(f) > 0$. In particular, the conjecture holds for germs of $\mathscr{A}_e$ codimension 1.

(6) Ayse Altintas Sharland gave three examples of $\mathscr{A}$-finite quasihomogeneous germs of corank 3 $(\mathbb{C}^3, 0) \to (\mathbb{C}^4, 0)$ with and was able to check that they satisfied the conjecture. For example,

$$f(x, y, z) = (y^2 + xz, x^5 + yz + xy^2, x^6 + y^3 + z^2, x^7 + x^4z + xz^2 + y^2z)$$

has $\mu_I = \mathscr{A}_e - \text{codim} = 18,967$, and another has $\mu_I = \mathscr{A}_e - \text{codim} = 127,295$. Sharland computed the $\mathscr{A}_e$ codimension using a method described below, and then calculated the image Milnor number using a remarkable formula of Toru Ohmoto, ([Ohm15]) proved using characteristic classes and Thom polynomials, which gives μ_I for a quasihomogeneous germ $(\mathbb{C}^3, 0) \to (\mathbb{C}^4, 0)$ as a rational function of weights and degrees. Formulae similar to Ohmoto's have subsequently been found for $n = 4$ and $n = 5$, by Irma Pallarés and Guillermo Peñafort, see [PPnS21], and can be used to check the conjecture for quasihomogeneous examples in these dimensions – the $\mathscr{A}_e$ codimension can be calculated on a computer using a method described in Subection 2.1.2 below. . Our joint paper [GM93] gave formulae (found by Victor Goryunov) for the ranks of the alternating homology groups of the multiple point spaces of a stable perturbation of a corank 1 map-germ; the sum of these ranks is the image Milnor number. It would be an interesting project to calculate μ_I using these formulae for the cases $n = 2, 3, 4$ and 5, and compare with the formulae of Ohomoto, Peñafort and Pallarés. In their recent survey paper [NnBPS], Nuño and Peñafort organise these formulae in an intriguing way: the formula for each n has n parts, with the k'th part taking the form $\dfrac{P_{n,k}}{k!}$, for $k = 2, \ldots n + 1$, where $P_{n,k}$ is a polynomial with integer coefficients. Are these n parts the ranks of the alternating cohomology of $D^2(f_t), \ldots, D^{n+1}(f_t)$?

2 Moving closer to the target

There are three problems with expression (1.3) for $T^1_{\mathscr{A}_e} f$.

(1) It is very hard to calculate directly, and in fact quite a lot of machinery was developed in the 1970s and 1980s by Mather, Gaffney and du Plessis to make the calculation possible.

(2) It has a "mixed module" structure, with $\theta(f)$ and $tf(\theta_n)$ being $\mathcal{O}_n$-modules, while θ_p is an $\mathcal{O}_p$-module. Every $\mathcal{O}_n$-module acquires an $\mathcal{O}_p$-module structure via f^*, so in particular $T^1_{\mathscr{A}_e} f$ is an $\mathcal{O}_p$-module. But this structure is hard to relate to the geometry of the image.

(3) $T^1_{\mathscr{A}_e} f$ determines how the singularity can be deformed, much like the Tjurina algebra $\dfrac{\mathcal{O}_n}{J_g + (g)}$ of an isolated hypersurface singularity $\{g = 0\}$. To understand the topology its Milnor fibre we need the jacobian algebra, $\mathcal{O}_n / J_g$. The Tjurina algebra is a quotient of the Milnor algebra. *Calculating $T^1_{\mathscr{A}_e} f$ as above, it does not seem to be the quotient of anything which will give us the required information.*

To overcome these difficulties, we look for a way to move $T^1_{\mathscr{A}_e} f$ into the target.

Lecture 2

2.1 Jacobian ideals

2.1.1 Notation

- Coordinates on the source $\mathbb{C}^n$ will always be denoted by $x_1, \ldots, x_n$, and sometimes we will denote the domain $\mathbb{C}^n$ by X.
- Coordinates on the target $\mathbb{C}^p$ will always be denoted by $y_1, \ldots, y_p$, and sometimes we will denote the target $\mathbb{C}^p$ by Y.
- Coordinates on the parameter space of an unfolding $F : (\mathbb{C}^n \times \mathbb{C}^d, S \times \{0\}) \to (\mathbb{C}^p \times \mathbb{C}^d, (0, 0))$ $\mathbb{C}^n$ will always be denoted by $u_1, \ldots, u_d$, and sometimes we will denote the parameter space $\mathbb{C}^d$ by U.
- If $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$, the *critical set* of f is the set $\Sigma := \{x \in \mathbb{C}^n : d_x f \text{ is not surjective}\}$ (or Σ_f when we want to specify which map we are talking about) , and we denote its image $f(\Sigma)$ by D. Note that when $n < p$ then $\Sigma = \mathbb{C}^n$ and D is the image of f.
- $\mathscr{O}_n$ is the ring of convergent power series in n complex variables, and thus the same as the ring of germs at 0 of holomorphic functions in n complex variables.
- Composition with f induces a homomorphism $f^* : \mathscr{O}_p \to \mathscr{O}_n$. We will also use f^* to denote the composite $\mathscr{O}_p \xrightarrow{f^*} \mathscr{O}_n \xrightarrow{q} \mathscr{O}_\Sigma$, where q is the passage to the quotient.
- The quotient homomorphism $\mathscr{O}_D \to \mathscr{O}_\Sigma$ induced by f^* will be denoted by $\overline{f^*}$.
- Given a function $g \in \mathbb{P}_p$, its *jacobian ideal* is the ideal J_g generated by its first order partial derivatives.

Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$, with $n+1 \geq p$. The case where $n + 1 = p$ is what interests us most here, but the case $n \geq p$ has a lot in common. For $\mathscr{A}$-finite f, Σ is $p-1$-dimensional, and $f|_\Sigma$ is finite and generically 1-to-1 – just like an $\mathscr{A}$-finite germ $(\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$.

For the moment, we treat the two cases together.

2.1.2 We put vector fields to work

To push $T^1_{\mathscr{A}_e} f$ into the target, we use the homomorphism $f^*(tg)$ originally suggested by Theo de Jong and Duco van Straten for the case $n + 1 = p$ (and used in [Mon91] to prove the conjecture for the case $n = 2$). It is defined by *differentiating the defining equation g of the discriminant or image of f with respect to vector fields in $\theta(f)$*. That is, each vector field along f, $\xi = \sum_i \alpha_i \partial/\partial y_i$, can be applied to g to give $f^*(tg)(\xi) := \sum_i \alpha_i(\partial g/\partial y_i \circ f)$ in $f^*(J_g)\,\mathscr{O}_\Sigma$. The α_i are elements of

$\mathcal{O}_n$, but we consider them in the quotient ring $\mathcal{O}_\Sigma$, so $f^*(tg)(\xi)$ is an $\mathcal{O}_\Sigma$- linear combination of the $(\partial g/\partial y_j)\circ f$.

At various points in the arguments which follow, we ought to distinguish between $f^*:\mathcal{O}_p \to \mathcal{O}_{\Sigma_f}$, $f^*:\mathcal{O}_p \to \mathcal{O}_n$ and $f^*:\mathcal{O}_D \to \mathcal{O}_{\Sigma_f}$. Mostly the difference is unimportant, but the last, with domain $\mathcal{O}_D$, will be denoted by $\overline{f^*}$. This becomes important when we consider preimages: preimages under $\overline{f^*}$ are different from preimages under f^*.

Theorem 2.1. *Let $f:(\mathbb{C}^n,0)\to(\mathbb{C}^p,0)$ be $\mathscr{A}$-finite, with $n+1\geq p\geq 2$ (but excluding the case $(n,p)=(1,2)$), and let g be a reduced defining equation for the discriminant D. Then $f^*(tg)$ passes to the quotient to give isomorphisms*

$$\frac{\theta(f)}{tf(\theta_n)} \to f^*(J_g)\,\mathcal{O}_\Sigma \tag{2.1}$$

$$\frac{\theta(f)}{T\mathscr{A}_e f} \to \frac{f^*(J_g)\,\mathcal{O}_\Sigma}{f^*(J_g\,\mathcal{O}_D)} \tag{2.2}$$

Remark 2.2. (1) Composition with f induces a monomorphism $\bar{f}^*:\mathcal{O}_D \to \mathcal{O}_\Sigma$, and $\bar{f}^*(J_g\,\mathcal{O}_D)$ is a subset (though not an ideal) of $\mathcal{O}_\Sigma$, while $\bar{f}^*(J_g)\,\mathcal{O}_\Sigma$ is the ideal in $\mathcal{O}_\Sigma$ that it generates. (Note that the image of f^* in $\mathcal{O}_\Sigma$ is the same as the image of $\overline{f^*}$ in $\mathcal{O}_\Sigma$. The difference only becomes meaningful when we consider preimages.) But, crucially for the practical value of the proposition, $\overline{f^*}(J_g)\,\mathcal{O}_\Sigma$ can also be thought of as an ideal of $\mathcal{O}_D$ – more precisely, it is the isomorphic image under $\overline{f^*}$ of an ideal in $\mathcal{O}_D$ (we explain this after the proof of the proposition) – so that the right hand side in (2.2) is isomorphic to a quotient of two ideals of $\mathcal{O}_D$. Writing it in this way, it becomes

$$\frac{\overline{f^*}^{-1}(\overline{f^*}(J_g)\,\mathcal{O}_\Sigma)}{J_g\,\mathcal{O}_D}. \tag{2.3}$$

Even though this formula looks discouragingly complicated, computing with it is particularly easy. For example, in addition to calculating f^*, Macaulay2 has a 'preimage' command which can be applied to $\overline{f^*}^{-1}$ and makes calculating both numerator and denominator of (2.3) straightforward.

(2) At two points, the proof of 2.1 involves the notion of the *depth* of a module over a commutative local ring. We will introduce it in a way that we hope will show its usefulness, and leave formal definition to an appendix.

2.1.3 The conductor ideal

When $R\subset S$ is an inclusion of rings, the *conductor* is the set (evidently an ideal) $\{r\in R: rS\subset R\}$. Similarly if $\phi: R\to S$ is a monomorphism of rings, the conductor ideal is $\{r\in R:\phi(r)S\subset\phi(R)\}$. It is obvious from the definition that the conductor is also an ideal of S. We denote it by $\mathscr{C}$. Its relevance to us here is

Lemma 2.3. *Let $\mathscr{C}$ denote the conductor of the monomorphism $\mathcal{O}_D \to \mathcal{O}_\Sigma$. Then*

(1) $J_g\,\mathcal{O}_D \subset \mathscr{C}$, *and* $V(J_g\,\mathcal{O}_D) = D_{sing} = V(\mathscr{C})$.

(2) The zero-locus of $\mathscr{C}$ *as an ideal in* $\mathcal{O}_\Sigma$ *is the set*

$$D_1^2(f) := \textit{closure of } \{x \in \mathbb{C}^n : \textit{there exists } x' \neq x \textit{ such that } f(x') = f(x)\}.$$

(3) When $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$, $\mathscr{C}$ *is a principal ideal in* $\mathcal{O}_n$, *and*

(4) ([Pie79]) *In this case, the quotient of* $(-1)^j \dfrac{\partial g}{\partial y_j}$ *by* $\dfrac{\partial(f_1, \ldots, \widehat{f_j}, \ldots f_{n+1})}{\partial(x_1, \ldots, x_n)}$ *is independent of* j, *and generates* $\mathscr{C}$ *as an ideal in* $\mathcal{O}_n$.

Proof of (1) and (2) (sketch) Any function coming from $\mathcal{O}_D$ must take the same value at pairs of points in Σ with the same image. The only way that multiplying any arbitrary function $a \in \mathcal{O}_\Sigma$ by $h \circ f$ can achieve this is that $h \circ f$ must vanish at all such points. In fact for $\mathscr{A}$-finite $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ this is also a sufficient condition. The set of points in D with more than one preimage is precisely the singular set of D. When $p = n+1$, but not when $n \geq p$, $\mathscr{C}$ is radical and $\mathscr{C} = I(D_{\text{sing}})$. In both cases $J_g\,\mathcal{O}_D \subset \mathscr{C}$. □

Exercise 2.4. *(i) The Whitney cusp map* $(x, y) \mapsto (x, y^3 + xy)$ *is stable, and hence so is* $f(w, x, y) = (w, x, y^3 + xy)$. *Find* Σ, D *and* $\mathscr{C}$ *and check that in this case* $\mathscr{C}$ *is not radical.*

(ii) Check that for the stable germ $(x, y) = (x, y^2, xy)$, $\mathscr{C}$ *is radical, and verify statement (4) of Lemma 2.3 for this case.*

Part (1) of the lemma justifies the statement in Remark 2.2 above that $f^*(J_g)\,\mathcal{O}_\Sigma$ is the isomorphic image of an ideal in $\mathcal{O}_D$.

Proof of Theorem 2.1

Step 1 For any point $x \in \Sigma$ and vector $v \in T_x\mathbb{C}^n$, $d_xf(v)$ is tangent to D. In case $n \geq p$ this can be checked at fold points, which are dense. In case $n + 1 = p$ it is obvious. In any case it follows that $f^*(tg)(tf(\xi)) = t(g \circ f)(\xi)$ vanishes on Σ. Thus $f^*(tg) : \theta(f) \to J_g\,\mathcal{O}_n$ passes to the quotient to define an epimorphism $\theta(f)/tf(\theta_n) \to J_g\,\mathcal{O}_\Sigma$.

Step 2 $\theta(f)/tf(\theta_n)$ has depth greater than 0. To see this, consider the exact sequence

$$\theta_n \longrightarrow \theta(f) \xrightarrow{tf} \frac{\theta(f)}{tf(\theta_n)} \longrightarrow 0 \tag{2.4}$$

where θ_n and $\theta(f)$ are free $\mathcal{O}_n$-modules of rank n and p respectively. When $n \geq p$, $\theta(f)/tf(\theta_n)$ has dimension $p - 1$ and hence is Cohen-Macaulay (i.e. its depth is equal to its dimension), by the theorem of Buchsbaum-Rim [BR64], quoted as Theorem 5.5 below. So, it has depth $p-1 > 0$. When

$p = n+1$, f is a finite mapping, which implies that tf is injective. Thus (2.4) is a free resolution of $\theta(f)/tf(\theta_n)$, which therefore has projective dimension, as $\mathcal{O}_n$-module, equal to 1, and hence depth $n-1>0$, by the Auslander–Buchsbaum Theorem (5.3 below) relating depth to projective dimension.

Step 3: Because $\theta(f)/tf(\theta_n)$ has positive depth, it has no submodule which is supported only at the origin. This follows immediately from the definition of depth.

Step 4: Let $K = \ker f^*(tg) : \theta(f)/tf(\theta_n) \to f^*(J_g)\,\mathcal{O}_\Sigma$. We show that K, if not zero, is supported only at 0. To do this we use the geometrical criterion for $\mathscr{A}$-finiteness, that f must be stable outside 0. So consider the case where f is stable. There is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Der}(-\log D) & \hookrightarrow & \theta_p & \xrightarrow{\overline{\omega f}} & \dfrac{\theta(f)}{tf(\theta_n)} & \longrightarrow & 0 \\ & & \Big\downarrow = & & \Big\downarrow = & & \Big\downarrow f^*(tg) & & \\ 0 & \longrightarrow & \mathrm{Der}(-\log D) & \hookrightarrow & \theta_p & \xrightarrow{tg} & J_g\,\mathcal{O}_\Sigma & \longrightarrow & 0 \end{array} \tag{2.5}$$

Here $\overline{\omega f}$ is just ωf followed by the projection to the quotient $\theta(f) \to \theta(f)/tf(\theta_n)$. And $\mathrm{Der}(-\log D)$ is the $\mathcal{O}_p$-submodule of θ_p consisting of germs of vector fields tangent to D at its smooth points. It is easy to show that that

$$\mathrm{Der}(-\log D) = \{\eta \in \theta_p : tg(\eta) \in (g)\},$$

which proves exactness of the second row at θ_p. That $\mathrm{Der}(-\log D)$ is the kernel of $\overline{\omega f}$ is just the statement that $\eta \in \theta_p$ is liftable via f (i.e. there exists $\xi \in \theta_n$ such that $tf(\xi) = \omega f(\eta)$) if and only if $\eta \in \mathrm{Der}(-\log D)$. This is well known (see e.g. [MNnB22, Proposition 8.8]). Surjectivity of $\overline{\omega f}$ is due to the infinitesimal stability of f, and surjectivity of tg follows by commutativity of the diagram, since $f^*(tg)$ is clearly surjective.

Given the exactness of the rows and the fact that the first two vertical arrows are isomorphisms, the kernel and cokernel of $f^*(tg)$ are equal to 0. This completes the proof that (2.1) is an isomorphism. That (2.2) is also an isomorphism follows, since the image of tg in $J_g\,\mathcal{O}_\Sigma$ is $J_g\,\mathcal{O}_D$. □

2.2 Expanding into the ambient $\mathbb{C}^p$

Javier Fernández, Juanjo Nuño and Guillermo Peñafort in [FdBNnBPnS19], tackle problem (3) in the list of at the start of Section 2 by expanding the quotient in (2.2) "into the ambient space" $\mathbb{C}^p$; they consider

$$\frac{f^{*-1}(f^*(J_g)\,\mathcal{O}_\Sigma)}{J_g}. \tag{2.6}$$

This is a quotient of two ideals of $\mathcal{O}_p$, whereas the quotient $\dfrac{\overline{f^*}^{-1}(f^*(J_g)\,\mathcal{O}_\Sigma)}{J_g\,\mathcal{O}_D}$, isomorphic to $T^1_{\mathscr{A}_e}f$, is a quotient of two ideals of $\mathcal{O}_D$. To simplify notation we write the top line in (2.6) as I_g.

Remark 2.5. We note that when g is quasihomogeneous, then $g \in J_g$, so that the natural epimorphism

$$\frac{f^{*-1}(f^*(J_g)\,\mathcal{O}_\Sigma)}{J_g} \to \frac{\overline{f^*}^{-1}(f^*(J_g)\,\mathcal{O}_\Sigma)}{J_g\,\mathcal{O}_D}$$

is an isomorphism. Thus $\dim_{\mathbb{C}} \dfrac{I_g}{J_g} \geq \mathscr{A}_e - \operatorname{codim}(f)$ with equality when f is quasihomogeneous.

Conjecture 2.6. (Fernández de Bobadilla, Nuño, Peñafort) *When $p = n+1$ and (n,p) are nice dimensions then $\dim_{\mathbb{C}} I_g/J_g = \mu_I(f)$.*

This conjecture implies the Mond conjecture.

Give an unfolding $F : (\mathbb{C}^n \times \mathbb{C}^d, (0,0)) \to (\mathbb{C}^p \times \mathbb{C}^d, (0,0))$ of f whose discriminant has equation G, there is a relative version of I_g/J_g, namely

$$\frac{I_G^{\mathrm{rel}}}{J_G^{\mathrm{rel}}} := \frac{F^{*-1}(F^*(J_G^{\mathrm{rel}})\,\mathcal{O}_{n+d}))}{J_G^{\mathrm{rel}}},$$

where $J_G^{\mathrm{rel}} = \left(\dfrac{\partial G}{\partial y_1}, \ldots, \dfrac{\partial G}{\partial y_p}\right)$ (i.e. we omit the partial derivatives with respect to the unfolding parameters). A difficulty which surfaces here is that it is not obvious that reducing $I_G^{\mathrm{rel}}/J_G^{\mathrm{rel}}$ modulo $\mathfrak{m}_d$ gives I_g/J_g, i.e. that

$$\frac{I_G^{\mathrm{rel}}/J_G^{\mathrm{rel}}}{(u_1,\ldots,u_d)I_G^{\mathrm{rel}}/J_G^{\mathrm{rel}}} = \frac{I_g}{J_g} \tag{2.7}$$

(or equivalently, that $\dfrac{I_G^{\mathrm{rel}}}{J_G^{\mathrm{rel}}} \otimes_{\mathcal{O}_{p+d}} \dfrac{\mathcal{O}_d}{\mathfrak{m}_d} = \dfrac{I_g}{J_g}$). Nevertheless, Nuño and Peñafort show in [NnBPS]

Lemma 2.7. *If F is a stable unfolding of an $\mathscr{A}$-finite map-germ $(\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$, then*

(1) (2.7) holds, and hence

(2) $\dfrac{I_G^{rel}}{J_G^{rel}}$ is a finitely generated $\mathcal{O}_d$-module of dimension d, with $\dim_{\mathbb{C}} \dfrac{I_g}{J_g}$ generators. □

Note that (2) follows from (1) by the Preparation Theorem and Nakayama's Lemma.

The truth or falsity of their conjecture rests on whether $I_G^{\mathrm{rel}}/J_G^{\mathrm{rel}}$ is a *free* module over $\mathcal{O}_d$. In the classical theory of isolated hypersurface singularities, freeness of the relative jacobian algebra means that a critical point with Milnor number μ splits on deformation into μ non-degenerate critical points (this is "conservation of multiplicity"), and the role of freeness would be similar here. To prove that $I_G^{\mathrm{rel}}/J_G^{\mathrm{rel}}$ is free over $\mathcal{O}_d$, it is necessary to show first that it is Cohen-Macaulay when considered as an $\mathcal{O}_{n+1+d}$-module. This still remains conjectural.

The notion of Cohen-Macaulay module is crucial in this story, and will surface repeatedly in what follows. My view on how to understand a new definition is "see what it does for you". I hope that in these lectures you will see how the Cohen-Macaulayness of certain modules associated to an unfolding implies conservation of multiplicity and allows us to calculate $\mu_I(f$ (or $\mu_\Delta(f)$ when $n \geq p$) and compare it with the $\mathscr{A}_e$-codimension of f.

The proof of Lemma 2.7(1) does not work for the case $n \geq p$, and so to deal with that, in Subsection 2.4 we end up making use of a different "expansion into the ambient space" with a different relative module, of a more traditional kind, for which it is obvious that it reduces appropriately modulo $\mathfrak{m}_d$.

As far as possible, we will continue to develop together the arguments for the conjecture, and for Theorem 1.9. This will help to show clearly where the difficulty for the conjecture is.

Lemma 2.8. *(1) If F is a stable unfolding then* $\dfrac{I_G^{rel}}{J_G^{rel}} = \dfrac{J_G + (G)}{J_G^{rel}}$.

(2) If in addition $G \in J_G$ then $\dfrac{I_G^{rel}}{J_G^{rel}} = \dfrac{J_G}{J_G^{rel}}$.

Proof For any unfolding F, $J_G^{\mathrm{rel}}\,\mathcal{O}_{\Sigma_F} = J_G\,\mathcal{O}_{\Sigma_F}$. For since D is the image of Σ_F, $G \circ F = 0$ on Σ_F, and thus $dG \circ dF = 0$ on Σ_F. Differentiating with respect to u_i, and writing the unfolding F as $(\tilde{f}(x,u),u)$, at points of Σ_F we therefore have

$$0 = \frac{\partial(G \circ F)}{\partial u_i} = \sum_{j=1}^{p}\left(\frac{\partial G}{\partial y_j} \circ F\right)\frac{\partial \tilde{f}_j}{\partial u_i} + \frac{\partial G}{\partial u_i} \circ F,$$

showing that $\dfrac{\partial G}{\partial u_i} \circ F \in J_G^{\mathrm{rel}}\,\mathcal{O}_{\Sigma_F}$.

Since F is stable, $J_G\,\mathcal{O}_{\Sigma_F} = J_G\,\mathcal{O}_{D_F}$, by Theorem 2.1. This implies

$$I_G^{\mathrm{rel}} = (F^*)^{-1}(J_G^{\mathrm{rel}}\,\mathcal{O}_{\Sigma_F}) = (F^*)^{-1}(J_G\,\mathcal{O}_{\Sigma_F}) = J_G + (G). \qquad \square$$

It will be useful in what follows to assume that $G \in J_G$. This can always be arranged; if G is quasihomogeneous then it is easy to show that $G \in J_G$, while in general, we can replace the stable unfolding F by $F \times \mathrm{id}_{\mathbb{C}}$ (which is still stable), and, denoting the extra unfolding parameter by t, replace G by $G_0 = e^t G$, and then use the fact that $G_0 = \partial G_0/\partial t \in J_{G_0}$. An equation G for which $G \in J_G$ is called a *good defining equation* for D.

To simplify our notation we write $\mathbb{C}^n = X, \mathbb{C}^p = Y$ and $\mathbb{C}^d = U$, and denote the projection $Y \times U \to U$ by π.

Lemma 2.9. *If F is a stable unfolding then* $\dfrac{J_G}{J_G^{rel}} \simeq \dfrac{\theta(\pi)}{t\pi(\ker\ dG)}$

Proof J_G is the image of $dG : \theta_{Y\times U} \to \mathcal{O}_{Y\times U}$, so

$$J_G \simeq \frac{\theta_{Y\times U}}{\ker dG}$$

and it follows that

$$\frac{J_G}{J_G^{\mathrm{rel}}} \simeq \frac{\theta_{Y\times U}}{\ker dG + \mathcal{O}_{Y\times U}\{\frac{\partial}{\partial y_1},\cdots,\frac{\partial}{\partial y_p}\}}.$$

Now $\theta_{Y\times U} = \mathcal{O}_{Y\times U}\{\frac{\partial}{\partial y_1},\cdots,\frac{\partial}{\partial y_p},\frac{\partial}{\partial u_1},\cdots,\frac{\partial}{\partial u_d}\}$, so cancelling the generators $\frac{\partial}{\partial y_j}$ we are left with

$$\frac{J_G}{J_G^{\mathrm{rel}}} \simeq \frac{\mathcal{O}_{Y\times U}\{\frac{\partial}{\partial u_1},\cdots,\frac{\partial}{\partial u_d}\}}{t\pi(\ker dG)} \qquad \square$$

– where for a vector field $\xi := \sum_j a_j(y,u)\partial/\partial y_j + \sum_i b_i(y,u)\partial/\partial u_i$, $\quad t\pi(\xi)$ is just $\sum_i b_i(y,u)\partial/\partial u_i$.

The lemma suggests that to show that J_G/J_G^{rel} is Cohen Macaulay, we need to know something about ker dG. First we say something about the slightly larger module of vector fields

$$\mathrm{Der}(-\log D) := \{\xi \in \theta_{p+d} : dG(\xi) \in (G)\}, \tag{2.8}$$

mentioned in Step 4 of the proof of Theorem 2.1. It is the set of ambient vector field germs which are tangent to D at its smooth points. Clearly $\ker(dG) \subset \mathrm{Der}(-\log D)$: $\ker(dG)$ is the set of vector fields tangent to *all* the level sets of G.

Lemma 2.10. *If $F : (\mathbb{C}^n \times \mathbb{C}^d, S \times \{0\}) \to (\mathbb{C}^p \times \mathbb{C}^d, (0,0))$ is a stable unfolding and either*

(i) $n+1 = p$ and f (and therefore also F) is a multi-germ of immersions, or

(ii) $n \geq p$ and (n,p) are nice dimensions, then

(1) Der$(-\log D)$ is a free $\mathcal{O}_{p+d}$–module on $p+d$ generators, and

(2) If χ is a vector field such that $dG(\chi) = G$ (so that G is a good defining equation for D) then Der$(-\log D)$ splits as a direct sum of $\ker(dG)$ and the set of $\mathcal{O}_{p+d}$-multiples of χ, so that

(3) in this case $\ker dG$ is a free $\mathcal{O}_{p+d}$– module on $p+d-1$ generators.

Proof (1) In both cases, $\mathrm{Der}(-\log D)$ is the kernel of $\overline{\omega F} : \theta_p \to \dfrac{\theta(F)}{tF(\theta_n)}$. When $s \geq t$ then, since $\dfrac{\theta(F)}{tF(\theta_n)}$ has depth $t-1$, as noted in Step 2 of the proof of Theorem 2.1, it has projective dimension 1, by the Auslander-Buchsbaum theorem (5.3 below). It follows that the kernel of $\overline{\omega F}$ is free. In case (ii), D must be a normal crossing divisor (NCD), since in a stable multi-germ of immersions, the images of the component immersions must meet in general position. It is well known that NCDs are free divisors. Kyogi Saito showed in [Sai80], where he introduced the notion of free divisor, that a hypersurface germ $B \subset \mathbb{C}^q$ is a free divisor if and only if $\mathrm{Der}(-\log B)$ is generated by precisely q vector field germs. This proves (1). I leave the easy proof of (2) as an exercise, and (3) follows from (1) and (2). $\square$

Exercise 2.11. K.Saito showed in [Sai80] that a hypersurface germ D at a point $x_0 \in \mathbb{C}^q$ is a free divisor if and only if there are germs of vector fields $\chi_1, \ldots, \chi_q \in \mathrm{Der}(-\log D)_{x_0}$ such that the determinant of the $q \times q$ matrix of their coefficients is a reduced equation for D. In this case, $\chi_1, \ldots, \chi_q$ form a basis for $\mathrm{Der}(-\log D)$. *To do* (i) Find a basis for $\mathrm{Der}(\log D)$ when D is the NCD $\{(x_1, \ldots, x_q) \in \mathbb{C}^q : x_1 \cdots x_q = 0\}$. (ii) Do the same when D is the NCD $\{(x_1, \ldots, x_q) \in \mathbb{C}^q : x_1 \cdots x_\ell = 0\}$ with $\ell < q$. (iii) Do the same when D is the plane curve germ $\{(x_1, x_2) \in \mathbb{C}^2 : x_1^p - x_2^q = 0\}$ with p, q coprime.

Theorem 2.12. *If $F : (\mathbb{C}^n \times \mathbb{C}^d, S \times \{0\}) \to (\mathbb{C}^p \times \mathbb{C}^d, (0,0))$ is a stable unfolding of an $\mathscr{A}$-finite germ, and either (i) $n + 1 = p$ and f (and therefore also F) is a multi-germ of immersions, or*

(ii) $n \geq p$ and (n, p) are nice dimensions, then

(1) $\dfrac{J_G}{J_G^{rel}}$ is a Cohen-Macaulay module of dimension $p + d - 1$, and

(2) $\dfrac{J_G}{J_G^{rel}}$ is a free module over $\mathcal{O}_d$.

Proof (1) By Lemma 2.10, $\ker dG$ is free of rank $p + d - 1$, and $\theta(\pi)$ is free of rank d, so $\dfrac{J_G}{J_G^{\mathrm{rel}}}$ has a presentation

$$\mathcal{O}_{p+d}^{p+d-1} \to \mathcal{O}_{p+d}^{d} \to \frac{J_G}{J_G^{\mathrm{rel}}} \to 0.$$

The theorem of Buchsbaum and Rim ([BR64]) already quoted in the proof of Theorem 2.1 says that if

$$\mathcal{O}^a \to \mathcal{O}^b \to M \to 0$$

is a presentation of an $\mathcal{O}$-module M, with $a \geq b$, then codim supp$(M) \leq a - b + 1$, and if equality holds then M is Cohen-Macaulay. To use this here, we have to show that codim supp$\dfrac{J_G}{J_G^{\mathrm{rel}}} = p$. We already know that it is no greater than p, so we have just to show that it is no less than p. In other words, we have to show that its dimension is no greater than d. This will follow if we can show that the projection $\pi : \mathbb{C}^p \times \mathbb{C}^d \to \mathbb{C}^d$ is finite-to-one on supp$\dfrac{J_G}{J_G^{\mathrm{rel}}}$. This is equivalent to showing that

$$\dim_{\mathbb{C}} \left(\frac{I_G^{\mathrm{rel}} / J_G^{\mathrm{rel}}}{\mathfrak{m}_d I_G^{\mathrm{rel}} / J_G^{\mathrm{rel}}} \right) < \infty.$$

In the case where $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ is a multi-germ of immersions, this follows from the fact

(Lemma 2.6(1)) that

$$\frac{I_G^{\rm rel}/J_G^{\rm rel}}{\mathfrak{m}_d I_G^{\rm rel}/J_G^{\rm rel}} \simeq \frac{I_g}{J_g}.$$

We postpone proof that π is finite on supp J_G/J_G^{rel} for the case $n \geq p$ until after introducing an alternative to I_g/J_g in the next section.

(2) Because π is finite on its support, $J_G/J_G^{\rm rel}$ is a finite module over $\mathcal{O}_d$ (see 5.6 below). Its depth as $\mathcal{O}_d$-module is the same as its depth as $\mathcal{O}_{p+d}$–module ([BH93, Exercise 1.2.26]), namely d (recall that $J_G/J_G^{\rm rel}$ is Cohen Macaulay, and thus its depth is equal to its dimension). By the Auslander Buchsbaum theorem (quoted as Theorem 5.3 below), its projective dimension is 0 i.e. it is free as an $\mathcal{O}_d$-module. □

2.3 Proof of Theorem 1.9 (leaving a small gap), and Conjecture 1.1 for multi-germs of immersions

Once a representative $F : X \times U \to Y \times U$ of F is chosen, J_G and $J_G^{\rm rel}$ extend uniquely to coherent sheaves of $\mathcal{O}_{Y\times U}$-modules on $Y \times U$. We continue to use J_G and $J_G^{\rm rel}$ to denote these sheaves, but take care now to specify base points, so the old ideals J_G and $J_G^{\rm rel}$ in $\mathcal{O}_{p+d} = \mathcal{O}_{\mathbb{C}^p\times\mathbb{C}^d,(0,0)}$ become $J_{G,(0,0)}$ and $J^{\rm rel}_{G,(0,0)}$. The quotient sheaf $\dfrac{J_G}{J_G^{\rm rel}}$ is likewise coherent, with stalk at $(0,0)$ equal to $\dfrac{J_{G,(0,0)}}{J^{\rm rel}_{G,(0,0)}}$.

To lighten the notation, we denote the coherent sheaf $J_G/J_G^{\rm rel}$ by M.

By the finiteness of $\dim_{\mathbb{C}} M_{(0,0}/\mathfrak{m}_d M_{(0,0)}$, π is a finite mapping on supp M, so the push forward to U of M is a coherent sheaf of $\mathcal{O}_U$-modules, denoted $\pi_*(M)$. As with any finite push-forward, for each $u \in U$ we have

$$\pi_*(M) = \bigoplus_{(y,u)} M_{(y,u)} \tag{2.9}$$

where the direct sum is over points $(y,u) \in \pi^{-1}(u) \cap \mathrm{supp} M$ – there are only a finite number of these. If the representative F is good, then $(0,0)$ is the only point of supp$M$ in $\pi^{-1}(0)$. Thus the stalk of $\pi_*(M)$ at 0 is $M_{(0,0)} = J_{G,(0,0)}/J^{\rm rel}_{G,(0,0}$. By Theorem 2.12, this is a free $\mathcal{O}_d$-module, with rank equal to $\dim_{\mathbb{C}} M_{(0,0)}/\mathfrak{m}_d M_{(0,0}$ by Nakayama's Lemma. Freeness is an open property of a coherent sheaf, so shrinking U if necessary, $\pi_*(M)$ is a free sheaf of $\mathcal{O}_U$-modules, whose rank everywhere is the same as the rank of $\pi_*(M)_{(0,0)}$. Thus

$$\dim_{\mathbb{C}} \left(M_{(0,0)}/\mathfrak{m}_d M_{(0,0)}\right) = \sum_y \dim_{\mathbb{C}} \left(M_{(y,u)}/\mathfrak{m}_{d,u} M_{(y,u)}\right) \tag{2.10}$$

(we attach a subindex u to $\mathfrak{m}_d$ on the right, because we are referring to the maximal ideal in $\mathcal{O}_{\mathbb{C}^d,u}$) As before, the sum here is over points $(y,u) \in \pi^{-1}(u) \cap \mathrm{supp}\ M$.

Let u be a parameter value outside the bifurcation set – i.e. such that f_u is stable. Let $g_u = G \circ i_u$, and let $D_u = g_u^{-1}(0)$. We divide the sum (2.10) into two parts: where $(y,u) \in D_u$, and where $(y,u) \notin D_u$.

For each point $y \notin D_u$,

$$\frac{M_{(y,u)}}{\mathfrak{m}_{d,u} M_{(y,u)}} = \frac{\mathcal{O}_{Y \times U,(y,u)}}{J^{\mathrm{rel}}_{G_{(y,u)}} + \mathfrak{m}_{U,u}} \simeq \frac{\mathcal{O}_{Y,y}}{J_{g_u}}$$

Thus

$$\sum_{y \notin D_u} \frac{M_{(y,u)}}{\mathfrak{m}_{d,u} M_{(y,u)}} = \sum_{y \notin D_u} \dim_{\mathbb{C}} \frac{\mathcal{O}_{Y,y}}{J_{g \circ i_u}} \tag{2.11}$$

The right hand side in (2.11) is μ_I (or μ_Δ in the case $n \geq p$), by Siersma's result, Theorem 1.7.

At each point $y \in D_u$, we claim $M_{(y,u)} = 0$. For we are assuming f_u is stable. In the nice dimensions, all stable germs are quasihomogeneous, and so by Lemma 2.15 (which we prove shortly!) $M_{(y,u)}/\mathfrak{m}_{d,u} M_{(y,u)} = T^1_{\mathscr{A}_e} f_u = 0$.

Thus,

$$\sum_y \dim_{\mathbb{C}} M_{(y,u)} = \sum_{y \notin D(f_u)} \dim_{\mathbb{C}} M_{(y,u)} = \begin{cases} \mu_\Delta & \text{if } n \geq p \\ \mu_I & \text{if } n+1 = p \text{ and } f \text{ is a multi-germ of immersions} \end{cases}$$ □

2.4 Damon's method

Jim Damon showed in [Dam91] how to calculate $\mathcal{A}_e$-codimension by what at first sight is a completely different method. If $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ has stable unfolding $F : (\mathbb{C}^n \times \mathbb{C}^d, S \times \{0\}) \to (\mathbb{C}^p \times \mathbb{C}^d, (0,0))$ then there is a commutative diagram (from which I omit the base-points)

$$\begin{array}{ccc} \mathbb{C}^n \times \mathbb{C}^d & \xrightarrow{F} & \mathbb{C}^p \times \mathbb{C}^d \\ {\scriptstyle j}\uparrow & & \uparrow{\scriptstyle i} \\ \mathbb{C}^n & \xrightarrow[f]{} & \mathbb{C}^p \end{array} \tag{2.12}$$

in which the vertical arrows are just inclusions $x \mapsto (x,0)$ and $y \mapsto (y,0)$. This is a fibre square: the $\mathbb{C}^n$ in the bottom left is the fibre product of the domain of F and the domain of i over their target $\mathbb{C}^p \times \mathbb{C}^d$, and thus the maps $\begin{array}{l} {\scriptstyle j}\uparrow \\ \quad \xrightarrow[f]{} \end{array}$ are determined by the maps $\begin{array}{r} \xrightarrow{F} \\ \uparrow{\scriptstyle i} \end{array}$ shared

Everything about f must therefore be calculable from information about F and i. Consider the diagram

$$\begin{array}{ccccccccc}
\text{col. }\# & & \mathbf{2} & & \mathbf{1} & & \mathbf{0} & & \\
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & t\pi(\mathrm{Der}(-\log D)) & \hookrightarrow & \theta(\pi) & \longrightarrow & 0 & & \\
 & & \uparrow{\scriptstyle t\pi} & & \uparrow{\scriptstyle t\pi} & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Der}(-\log D) & \hookrightarrow & \theta_{Y\times U} & \xrightarrow{\omega F} & \dfrac{\theta(F)}{tF(\theta_{X\times U})} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow{\scriptstyle \simeq} & & \\
0 & \longrightarrow & \mathrm{Der}(-\log D)/U & \hookrightarrow & \theta_{Y\times U/U} & \xrightarrow{\omega F} & \dfrac{\theta(F/U)}{tF(\theta_{X\times U/U})} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{2.13}$$

in which $\theta_{Y\times U/U}$ means vector fields on $Y \times U$ with zero component in the U direction, and similarly $\theta(F/U)$ and $\mathrm{Der}(-\log D)/U$. A diagram chase shows

$$\frac{\theta(\pi)}{t\pi(\mathrm{Der}(-\log D))} \simeq \frac{\theta(F/U)}{tF(\theta_{X\times U/U}) + \omega F(\theta_{Y\times U/U})} \tag{2.14}$$

(that is, the first homology of the top complex is isomorphic to the 0'th homology of the bottom complex). The right hand side here is the relative module $T^{1\ \mathrm{rel}}_{\mathscr{A}_e}F$, for which we know that

$$\frac{T^{1\ \mathrm{rel}}_{\mathscr{A}_e}F}{\mathfrak{m}_d T^{1\ \mathrm{rel}}_{\mathscr{A}_e}F} \simeq T^1_{\mathscr{A}_e}f \tag{2.15}$$

On the other hand, since the left hand side in (2.14) is isomorphic to

$$\frac{\theta_{p+d}}{\mathcal{O}_{p+d}\{\partial/\partial y_1, \ldots, \partial/\partial y_p\} + \mathrm{Der}(-\log D)}$$

it follows that

$$M_{(0,0)}/\mathfrak{m}_d M_{(0,0)} = \frac{\theta_{p+d}}{\mathcal{O}_{p+d}\{\partial/\partial y_1, \ldots, \partial/\partial y_p\} + \mathrm{Der}(-\log D) + \mathfrak{m}_d\theta_{p+d}} \tag{2.16}$$

By dividing by $\mathfrak{m}_d\theta_{p+d}$ we are restricting everything to $\mathbb{C}^p \times \{0\}$, so (2.16) can be rewritten as

$$\frac{\theta(i)}{\mathcal{O}_p\{\partial/\partial y_1, \ldots, \partial/\partial y_p\} + i^*(\mathrm{Der}(-\log D))}. \tag{2.17}$$

where i is the inclusion in (2.12). Finally, we note that $\mathcal{O}_p\{\partial/\partial y_1, \ldots, \partial/\partial y_p\} = ti(\theta_p)$, so that (2.17) becomes

$$\frac{\theta(i)}{ti(\theta_p) + i^*(\mathrm{Der}(-\log D))} \tag{2.18}$$

It follows that the right hand sides in (2.15) and (2.18) are isomorphic:

$$T^1_{\mathscr{A}_e} f \simeq \frac{\theta(i)}{ti(\theta_p) + i^*(\mathrm{Der}(-\log D))}. \tag{2.19}$$

Damon showed in [Dam91] that (2.19) holds for *any* germ f obtained by transverse fibre product of i and F, with F stable but not necessarily an unfolding, and i not necessarily an immersion.

Exercise 2.13. Carry out the diagram chase to obtain the isomorphism (2.14) from diagram (2.13), and, for the class of $\sum_i \alpha_i \dfrac{\partial}{\partial u_i}$ in $\dfrac{\theta(\pi)}{t\pi(\mathrm{Der}(-\log D))}$, find an explicit expression for its image in $\dfrac{\theta(F/U)}{tF(\theta_{X\times U/U}) + \omega F(\theta_{Y\times U/U})}$.

2.5 We apply Damon's method to fill the gap in the proof of Theorem 1.9

Recall that

$$\frac{I_G^{\mathrm{rel}}}{J_G^{\mathrm{rel}}} \simeq \frac{J_G}{J_G^{\mathrm{rel}}} \simeq \frac{\theta(\pi)}{t\pi(\ker dG)}.$$

Applying the argument of the last paragraph to $\theta(\pi)/t\pi(\ker dG)$, we get

$$\frac{\theta(\pi)}{t\pi(\ker dG)} \Big/ \mathfrak{m}_d \left(\frac{\theta(\pi)}{t\pi(\ker dG)}\right) \simeq \frac{\theta(i)}{ti(\theta_p) + i^*(\ker dG)} \tag{2.20}$$

where $i : (\mathbb{C}^p, 0) \to (\mathbb{C}^{p+d}, 0)$ induces f from the stable unfolding F by transverse fibre product, as described in the last subsection. Thus, we have a replacement for the conclusion of Lemma 2.7:

Lemma 2.14.

$$\frac{I_G^{rel}/J_G^{rel}}{\mathfrak{m}_d(I_G^{rel}/J_G^{rel})} \simeq \frac{\theta(i)}{ti(\theta_p) + i^*(\ker dG)}. \qquad \square$$

There is an obvious epimorphism

$$\frac{\theta(i)}{ti(\theta_p) + i^*(\ker dG)} \to \frac{\theta(i)}{ti(\theta_p) + i^*(\mathrm{Der}(-\log D))}. \tag{2.21}$$

Lemma 2.15. *When f is quasihomogeneous then* (2.21) *is an isomorphism.*

Proof If f is quasihomogeneous then we can choose its stable unfolding F to be quasihomogeneous also, though not necessarily with all weights positive. But in any case there is an Euler vector field χ_E in $\mathrm{Der}(-\log D)$ for which $tG(\chi_E) = G$, and thus

$$\mathrm{Der}(-\log D) = \ker dG + \mathcal{O}_{p+d}\, \chi_E.$$

If χ_e is the corresponding Euler vector field on $\mathbb{C}^p$ then $ti(\chi_e) = \chi_E \circ i$ so that

$$ti(\theta_p) + i^*(\mathrm{Der}(-\log D)) = ti(\theta_p) + i^*\big((\ker dG). \qquad \square$$

Lemma 2.16. *If F is a stable unfolding of an $\mathscr{A}$-finite germ $f:(\mathbb{C}^n,S)\to(\mathbb{C}^p,0)$, with $n\geq p$ or $n+1=p$, and (n,p) are nice dimensions, then $\dfrac{J_G}{J_G^{\text{rel}}}$ is finite over $\mathcal{O}_d$*

Proof We have just to show that $\left(\text{supp}\,\dfrac{J_G}{J_G^{\text{rel}}}\right)\cap\mathbb{C}^p\times\{0\}=\{(0,0)\}$. Because f is $\mathscr{A}$-finite, f is stable outside 0, so $\left(\text{supp}\,T^1_{\mathscr{A}_e}{}^{\text{rel}}F\right)\cap\mathbb{C}^p\times\{0\}=\{(0,0)\}$. Since we are in the nice dimensions, at a stable point y of f, f is quasihomogeneous with respect to suitable coordinates, so by Lemma 2.15, $(y,0)\notin\text{supp}\dfrac{J_G}{J_G^{\text{rel}}}$. Hence $(0,0)$ is the only point in supp $\left(\dfrac{J_G}{J_G^{\text{rel}}}\right)\cap\mathbb{C}^p\times\{0\}$. □

This completes the proof of Theorem 1.9.

3 The hard part of Conjecture 1.1 still remains unproved – but is surely true

If $f:(\mathbb{C}^n,S)\to(\mathbb{C}^{n+1},0)$ is $\mathscr{A}$-finite but not a multi-germ of immersions, then the image, D, of a stable unfolding is *not* a free divisor, so the argument of Theorem 2.12 fails. For example, the simplest non-immersive stable germ is the parameterisation of the Whitney umbrella,

$$f(x_1,x_2)=(x_1,x_2^2,x_1x_2)$$

Exercise 3.1. *Find and equation for D in this case, and go on to find generators of $Der(-\log D)$ for this germ, and check that D is not a free divisor (see Exercise 2.11). It is easiest to start by finding generators for the submodule* $\ker dg$*, where g is a reduced defining equation for D.*

Nevertheless, the fact that the conjecture holds in all of the examples we know means that in every case multiplicity *is* conserved, which, in turn, means that the module J_G/J_G^{rel} *must be* Cohen-Macaulay, even though no theorem that we are aware of says that it should be. So what can be done? I referred above to the proof of Cohen-Macaulayness of $I_G^{\text{rel}}/J_G^{\text{rel}}$ as "numerological". It is based on the theorem of Buchsbaum-Rim, that gives a sufficient numerical condition, the codimension of the support of the cokernel, M, of an $\mathcal{O}$-linear map $\mathcal{O}^a\to\mathcal{O}^b$, for M to be Cohen-Macaulay. We applied it in the proof of Theorem 2.12, arguing that when D is a free divisor and G is a good equation then $t\pi(\ker dG)$ has $p+d-1$ generators and so $\theta(\pi)/t\pi(\ker dG)$ has presentation

$$\mathcal{O}^{p+d-1}\to\mathcal{O}^d\to\frac{\theta(\pi)}{t\pi(\ker dG)}\to 0.$$

When $\ker dG$ needs more than $p+d-1$ generators (i.e. more than $n+d$ in the case $p=n+1$), the Buchsbaum-Rim theorem would require the support of $\dfrac{J_G}{J_G^{\text{rel}}}$ to have codimension greater than p, which it does not have. Nevertheless, for quasihomogeneous germs, the conjectured equality $\mu_I=\mathscr{A}_e$-codimension can *only* hold if $M:=J_G/J_G^{\text{rel}}$ is Cohen-Macaulay. If $\text{depth}_{\mathcal{O}_{n+d+1}}M$ were less than d then M would not be free as $\mathcal{O}_d$-module, from which it would follow that $\mu_I(f) < \mathscr{A}_e$–codim(f). Thus, all of the examples of quasihomogeneous germs satisfying the conjecture also support the conjecture that J_G^{rel}/J_G *is* Cohen-Macaulay.

4 Speculations

To overcome the difficulty, two possibilities suggest themselves.

The first is to seek a presentation of $\dfrac{J_G}{J_G^{\mathrm{rel}}}$, in which $\ker dG$ is replaced by a submodule with only $n+d$ generators.

The second is to return to the definition of Cohen-Macaulay module, and look for other reasons why $\dfrac{J_G}{J_G^{\mathrm{rel}}}$ might be Cohen-Macaulay. This is what we now do, briefly and inconclusively.

Proposition 4.1. *Suppose that* $F : (\mathbb{C}^n \times \mathbb{C}^d, S \times 0) \to (\mathbb{C}^{n+1} \times \mathbb{C}^d, (0,0)$ *is an unfolding of* $\mathscr{A}$*-finite germ* f*, that* G *is a reduced equation for the image* D *of* F*, and that* J_G/J_G^{rel} *has dimension* d*. Then* J_G/J_G^{rel} *has depth* d *if and only if the unfolding parameters* $u_1, \ldots, u_d$ *form a regular sequence on* M*.*

Proof $M/(u_1, \ldots, u_d)M$ is a finitely generated module over the 0-dimensional ring $\mathcal{O}_d/\mathfrak{m}_d = \mathbb{C}$, so its dimension is 0, and Proposition 5.4(1) applies. □

So to prove the conjecture one could try directly to show that $u_1, \ldots, u_d$ form a regular sequence on J_G/J_G^{rel}. If this is true, then u_1 is regular on $J_G/J_G^{\mathrm{rel}} \otimes (\mathcal{O}_d/(u_2, \ldots, u_d)$ so as an opening step it would be sensible to try to seek reasons why in a 1-parameter unfolding F, whose image has equation G, the parameter might be regular on $I_G^{\mathrm{rel}}/J_G^{\mathrm{rel}}$. Perhaps the fact that the unfolding is not trivial can be used in some way to show that the unfolding parameter is regular. Or could the semi-universal property of a versal unfolding play a role? It may be that to focus on the numerology is a distraction.

4.1 More on non-linear sections of images and discriminants

In Subsection 2.4 we thought of the map f and its deformations in a different way: a map $\mathscr{A}$-equivalent to f is obtained from a stable map F by means of a fibre diagram (from which we omit base points)

$$
\begin{array}{ccc}
\mathbb{C}^N & \xrightarrow{F} & \mathbb{C}^P \\
{\scriptstyle \mathrm{pr}_1}\uparrow & & \uparrow{\scriptstyle i} \\
\mathbb{C}^N \times_{\mathbb{C}^P} \mathbb{C}^p & \xrightarrow{\mathrm{pr}_2} & \mathbb{C}^p \\
{\scriptstyle \simeq}\uparrow & \nearrow{\scriptstyle f} & \\
\mathbb{C}^n & &
\end{array}
\tag{4.1}
$$

where i can be any germ (not necessarily an immersion) which is transverse to F, so that the fibre product is non-singular, and $N - P = n - p$, so that the fibre product has dimension n. Writing D_f and D_F for the images or discriminants of f and F, we have $D_f = i^{-1}(D_F)$. A perturbation f_t of f is obtained by perturbing the map i to i_t; then f_t is stable if and only if i_t is transverse to the distribution $\mathrm{Der}(-\log D_F)$ at 0. This is simply because

$$T^1_{\mathscr{A}_e} f \simeq \frac{\theta(i)}{ti(\theta_p) + i^*(\mathrm{Der}(-\log D_F)}, \tag{4.2}$$

as we saw in Subsection 2.4, and because a simple application of Nakayama's lemma shows that the RHS in (4.2) is 0 if and only if

$$\frac{T_0\mathbb{C}^P}{d_0 i(T_0\mathbb{C}^p) + T_0^{\log} D_F} = 0, \tag{4.3}$$

where $T_0^{\log} D_F$ is the "logarithmic tangent space to D_F at 0",

$$T_0^{\log} D_F := \{\chi(0) : \chi \in \mathrm{Der}(-\log D)\}.$$

Exercise 4.2. Prove that (4.2) is 0 if and only if (4.3) holds .

Damon generalised this idea by simply considering the right hand side of the diagram (4.1); that is, we fix some hypersurface D of $\mathbb{C}^P$ and consider its preimage $i^{-1}(D)$ under a map $i : \mathbb{C}^p \to \mathbb{C}^P$. By perturbing i to a nearby map i_t which is logarithmically transverse to D, we obtain $i_t^{-1}(D)$, which Damon calls a "singular Milnor fibre" of $i^{-1}(D)$.

In such a situation, following Jim Damon, we can define a T^1 in a natural way, as the quotient in (4.2), and consider the homology of the singular Milnor fibre. The argument sketched in the proof of Theorem 1.7 shows that the singular Milnor fibre has the homotopy type of a wedge of spheres of dimension $p-1$, and the general conjecture is that the number of these spheres is the dimension of

$$\frac{\theta(i)}{ti(\theta_p) + i^*(kerdG)}, \tag{4.4}$$

where G is the equation of the hypersurface D. In the literature $\ker dG$ is often denoted by $\mathrm{Der}(-\log G)$ to emphasise its relation with $\mathrm{Der}(-\log D)$. Bill Bruce, Victor Goryunov, Jim Damon and others have studied "matrix singularities", where $\mathbb{C}^P$ is the (linear) space of $m \times m$ matrices, or symmetric matrices, or skew-symmetric matrices, and D is the hypersurface of singular matrices.

In all of these cases, the singular locus of D is Cohen-Macaulay, in the sense that $\mathcal{O}_P/J_G$ is a Cohen-Macaulay ring – as it is when D is a free divisor.

Theorem 4.3. ([GM05]) *Let $D \subset \mathbb{C}^P$ be a hypersurface, with defining equation G, such that $\mathcal{O}_P/J_G$ is Cohen Macaulay, and of codimension m_0 in $\mathbb{C}^P$. Suppose that $i : (\mathbb{C}^p, 0) \to (\mathbb{C}^P, 0)$ is logarithmically transverse to D outside* 0 *and $p = m_0$ or $p = m_0 - 1$. Then the singular Milnor fibre of $i^{-1}(D)$ has the homotopy type of a wedge of $\mu_\Delta(i)$ spheres of dimension $p-1$, where $\mu_\Delta(i)$ is the vector space dimension of* (4.4)*.*

In [Gor21, Conjecture 6.9], Goryunov makes the following conjecture:

Conjecture 4.4. *Let $i : (\mathbb{C}^p, 0) \to Mat_n$ be a germ of any of the three types of matrix families. Assume that p is at least the codimension of the singular locus of the discriminant D in Mat_n, and i is logarithmically transverse to D outside* 0*. Then $\mu_\Delta(M)$ is equal to the vector space dimension of* (4.4)*.*

Note that when $m_0 = 2$, then D is a free divisor and the arguments used above prove the conjectured equality.

4.2 Lie groups of equivalences

Given a subvariety $V \subset \mathbb{C}^P$, Jim Damon introduced the subgroup $\mathscr{K}_V$ of the group $\mathscr{K}$ acting on the space of germs $(\mathbb{C}^p, 0) \to (\mathbb{C}^P, 0)$. Two map-germs $i_0, i_1 : (\mathbb{C}^p, 0) \to (\mathbb{C}^P, 0)$ are $\mathscr{K}_V$-equivalent if there exist diffeomorphisms Φ of $(\mathbb{C}^p \times \mathbb{C}^P, (0, 0))$ and φ of $(\mathbb{C}^p, 0)$ such that

(1) Φ maps $\pi_1 \circ \Phi = \varphi \circ \pi_1$, (i.e. Φ lifts φ),

(2) Φ maps $\mathbb{C}^p \times V$ to itself (i.e. Φ preserves V)

(3) Φ induces a diffeomorphism graph$(i_0) \to$ graph(i_1).

He shows in [Dam91] that the tangent space to the $\mathscr{K}_V$-orbit of i in the space of germs, $T\mathscr{K}_V i$, is

$$ti(\mathfrak{m}_p\theta_p) + i^*(\mathrm{Der}(-\log V)),$$

and the extended tangent space $T\mathscr{K}_{V,e} i$ is

$$ti(\theta_p) + i^*(\mathrm{Der}(-\log V).$$

Changing condition (2) in the definition above to
(2') $G \circ \pi_2 \circ \Phi = \Gamma \circ \pi_2$
gives the group $\mathscr{K}_G$, and the tangent space and extended tangent space for $\mathscr{K}_G$ are

$$ti(\mathfrak{m}_p\theta_p) + i^*(\mathrm{Der}(-\log G)) \quad \text{and} \quad ti(\theta_p) + i^*(\mathrm{Der}(-\log G)$$

respectively.

Could the fact that the T^1, and the conjectured formula for μ_I and μ_Δ, can all be decribed in terms of Lie groups of equivalences provide the missing ingredient needed to prove the conjectures? Perhaps the fact that the modules involved are tangent spaces and quotients of tangent spaces gives them special properties which imply conservation of multiplicity.

5 Appendix: Depth and Cohen-Macaulay modules

First, we consider the notion of *depth*. Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$, and let M be a finitely generated R module. An element $a \in \mathfrak{m}$ is said to be *regular on M* if multiplication by a defines an injective map $M \to M$. A sequence $a_1, \ldots, a_r$ of elements of $\mathfrak{m}$ is a *regular sequence* on M if for each j, a_j is regular on $M/(a_1, \ldots, a_{j-1})M$. The *depth* of M is the length of a maximal regular sequence – in other words, a regular sequence $a_1, \ldots, a_r$ such that there is no element in $\mathfrak{m}$ which is regular on $M/(a_1, \ldots, a_r)M$. It is well known that all maximal regular sequences have the same length.

Exercise 5.1. *Prove, directly from the definition, that a module with depth d greater than 0 cannot contain a submodule with depth less than d. This fact is used in the proof of Theorem 2.1.*

Proposition 5.2. *For any R-module M, depth $M \leq$ dim M* □

There are a number of different but equivalent definitions of dimension, but for our purposes the simplest, in the case of $\mathcal{O}_p$ modules which interest us, is that dim M is the dimension of the set of zeroes of the annihilator ideal of M, $\{a \in \mathcal{O}_p : am = 0 \text{ for all } m \in M\}$. This is equal to the dimension of the support of the coherent sheaf $\mathscr{M}$ which extends M to a neighbourhood of $0 \in \mathbb{C}^p$.

An R-module is *Cohen-Macaulay* if its depth is equal to its dimension. Every finitely generated module over the ring $\mathcal{O}_p$ has a finite free resolution; this property in fact characterises all regular local rings. That is, for every $\mathcal{O}_p$ module M, there exists a finite exact sequence

$$0 \longrightarrow \mathcal{O}_p^{r_\ell} \xrightarrow{A_\ell} \mathcal{O}_p^{r_{\ell-1}} \xrightarrow{A_{\ell-1}} \cdots \longrightarrow \mathcal{O}_p^{r_1} \xrightarrow{A_1} \mathcal{O}_p^{r_0} \longrightarrow M \longrightarrow 0\ .$$

A free resolution is *minimal* if every entry in each of the matrices A_j lies in the maximal ideal $\mathfrak{m}$, in which case the length, ℓ, of the resolution is the minimum possible, and is known as the *projective dimension* of M as $\mathcal{O}_p$-module and denoted by $\mathrm{pd}_{\mathcal{O}_p}(M)$.

Proposition 5.3. (The Auslander-Buchsbaum Theorem) see e.g. [BH93, Theorem 1.3.3]*Let R be a Noetherian local ring and M a finitely generated R-module. Then*

$$pd_R M + depth_R M = depth_R R.$$ □

A regular local ring such as $\mathcal{O}_p$ has depth equal to its dimension, so for an $\mathcal{O}_p$-module the formula reads

$$\mathrm{pd}_{\mathcal{O}_p}(M) + \mathrm{depth}_{\mathcal{O}_p} M = p.$$

In particular, an $\mathcal{O}_p$-module with depth p must be free.

We use the relation between depth and projective dimension in proving the quotient of ideals theorem, 2.1, in proving that the discriminant of a stable map-germ $(\mathbb{C}^N, S) \to (\mathbb{C}^P, 0)$ is a free divisor (Lemma 2.10), and in proving that J_G/J_G^{rel} is a free module over the base space of a stable unfolding (Theorem 2.12).

Proposition 5.4. (e.g. [BH93, Theorems 2.1.2, 2.1.3]) *Let M be a Cohen-Macaulay $\mathcal{O}_p$-module, and $a_1, \ldots, a_r \in \mathfrak{m}_p$. Then*

(1) $a_1, \ldots, a_r$ is a regular sequence on M if and only if dim $M/(a_1, \ldots, a_r)M =$ dim $M - r$, and

(2) in this case $M/(a_1, \ldots, a_r)M$ is Cohen Macaulay.

Theorem 5.5. ([BR64, Corollary 2.7])*Let R be a Cohen-Macaulay local ring, and M an R-module with presentation*

$$R^m \to R^n \to M \to 0,$$

with $m \geq n$. Then codim supp $M \leq m - n + 1$ and if they are equal then M is a Cohen-Macaulay module. □

5.0.1 Finiteness: The Preparation Theorem and Nakayama's Lemma

We have used the following result in .

Theorem 5.6. *Suppose that M is a finitely generated $\mathcal{O}_n$-module, and that $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is analytic. Then*

(1) under the homomorphism $f^* : \mathcal{O}_p \to \mathcal{O}_n$ through which M becomes an $\mathcal{O}_p$-module, M is finitely generated over $\mathcal{O}_p$ if and only if $\dim_{\mathbb{C}} M/f^*(\mathfrak{m}_p)M < \infty$,

and

(2) in this case

elements $m_1, \ldots, m_r \in M$ generate M over $\mathcal{O}_p$ if and only if their classes in $M/f^*(\mathfrak{m}_p)M$ generate it as $\mathbb{C}$-vector space.

Here (1) is the Preparation Theorem in the form given to it by John Mather. In its traditional form it is due to Weierstrass. Bernard Malgrange proved a C^∞ version of the same statement when $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ is smooth. And (2) is a simple consequence of (1), by Nakayama's lemma.

Exercise 5.7. Use Nakayama's lemma to prove (2) above.

□

References

[BH93] Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993. MR MR1251956 (95h:13020)

[BR64] D. A. Buchsbaum and D. S. Rim, *A generalized Koszul complex. II. Depth and multiplicity*, Trans. Amer. Math. Soc. **111** (1964), 197–224. MR 0159860

[Dam91] James Damon, *$\mathcal{A}$-equivalence and the equivalence of sections of images and discriminants*, Singularity theory and its applications, Part I (Coventry, 1988/1989), Lecture Notes in Math., vol. 1462, Springer, Berlin, 1991, pp. 93–121. MR 1129027 (92m:32057)

[dJvS91] T. de Jong and D. van Straten, *Disentanglements*, Singularity theory and its applications, Part I (Coventry, 1988/1989), Lecture Notes in Math., vol. 1462, Springer, Berlin, 1991, pp. 199–211. MR 1129033 (93a:14003)

[DM91] James Damon and David Mond, *$\mathcal{A}$-codimension and the vanishing topology of discriminants*, Invent. Math. **106** (1991), no. 2, 217–242. MR MR1128213 (92m:58011)

[FdBNnBPnS19] J. Fernández de Bobadilla, J. J. Nuño Ballesteros, and G. Peñafort Sanchis, *A Jacobian module for disentanglements and applications to Mond's conjecture*, Rev. Mat. Complut. **32** (2019), no. 2, 395–418. MR 3942922

[GCNB23] R. Giménez Conejero and J.J. Nuño-Ballesteros, *A weak version of the mond conjecture*, Collectanea Mathematica **1** (2023), https://link.springer.com/article/10.1007/s13348–023–00411–x.

[GM93] Victor Goryunov and David Mond, *Vanishing cohomology of singularities of mappings*, Compositio Math. **89** (1993), no. 1, 45–80. MR 1248891 (94k:32058)

[GM05] V. Goryunov and D. Mond, *Tjurina and Milnor numbers of matrix singularities*, J. London Math. Soc. (2) **72** (2005), no. 1, 205–224. MR 2145736

[Gor91] Victor Goryunov, *Monodromy of the image of the mapping* $\mathbf{C}^2 \to \mathbf{C}^3$, Funktsional. Anal. i Prilozhen. **25** (1991), no. 3, 12–18, 95. MR 1139870 (92m:32060)

[Gor21] ———, *Vanishing cycles of matrix singularities*, J. Lond. Math. Soc. (2) **103** (2021), no. 3, 991–1015. MR 4245828

[HK99] Kevin Houston and Neil Kirk, *On the classification and geometry of corank* 1 *map-germs from three-space to four-space*, Singularity theory (Liverpool, 1996), London

Math. Soc. Lecture Note Ser., vol. 263, Cambridge Univ. Press, Cambridge, 1999, pp. xxii, 325–351. MR 1709361 (2000h:58068)

[Hou98] Kevin Houston, *On singularities of folding maps and augmentations*, Math. Scand. **82** (1998), no. 2, 191–206. MR 1646521 (99k:32070)

[Mar77] Jean Martinet, *Singularités des fonctions et applications différentiables*, Pontifíia Universidade Católica do Rio de Janeiro, Rio de Janeiro, 1977, Deuxième édition corrigée, Monografias de Matemática da PUC/RJ, No. 1. [Mathematical Monographs of the PUC/RJ, No. 1]. MR 0464292 (57 #4225)

[Mar82] ______, *Singularities of smooth functions and maps*, London Mathematical Society Lecture Note Series, vol. 58, Cambridge University Press, Cambridge, 1982, Translated from the French by Carl P. Simon. MR 671585 (83i:58018)

[Mat71] J. N. Mather, *Stability of C^∞ mappings. VI: The nice dimensions*, Proceedings of Liverpool Singularities-Symposium, I (1969/70) (Berlin), Springer, 1971, pp. 207–253. Lecture Notes in Math., Vol. 192. MR 0293670 (45 #2747)

[MNnB22] David Mond and Juan José Nuño Ballesteros, *Singularities of mappings*, Handbook of geometry and topology of singularities III, Springer, Cham, [2022] ©2022, pp. 81–144. MR 4461009

[Mon91] David Mond, *Vanishing cycles for analytic maps*, Singularity theory and its applications, Part I (Coventry, 1988/1989), Lecture Notes in Math., vol. 1462, Springer, Berlin, 1991, pp. 221–234. MR 1129035 (93a:32054)

[Mon95] ______, *Looking at bent wires—$\mathcal{A}_e$-codimension and the vanishing topology of parametrized curve singularities*, Math. Proc. Cambridge Philos. Soc. **117** (1995), no. 2, 213–222. MR 1307076 (95m:58017)

[NnBPS] Juan José Nuño Ballesteros and Guillermo Peñafort-Sanchis, *30 years of the mond conjecture.*

[Ohm15] Toru Ohmoto, *Singularities and characteristic classes for differentiable maps*, Singularities, São Carlos, 2012, Advanced studies in pure mathematics, Math. Soc. Japan, 2015, pp. 1–58.

[Pie79] Ragni Piene, *Ideals associated to a desingularization*, Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978), Lecture Notes in Math., vol. 732, Springer, Berlin, 1979, pp. 503–517. MR 555713 (81a:14001)

[PPnS21] Irma Pallarés and Guillermo Peñafort Sanchis, *Image Milnor number formulas for weighted-homogeneous map-germs*, Results Math. **76** (2021), no. 3, Paper No. 152, 23. MR 4281274

[Sai80] Kyoji Saito, *Theory of logarithmic differential forms and logarithmic vector fields*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), no. 2, 265–291. MR MR586450 (83h:32023)

[Sie91] Dirk Siersma, *Vanishing cycles and special fibres*, Singularity theory and its applications, Part I (Coventry, 1988/1989), Lecture Notes in Math., vol. 1462, Springer, Berlin, 1991, pp. 292–301. MR 1129039 (92j:32129)

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